# QUANTUM GROUP "COVERINGS" of $SL(2,\mathbb{C})$

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- $SL_q(2,\mathbb{C})$ at roots of unity $\Rightarrow$
  1) is a "q-group" covering of $SL(2)$
  2) has "q-comodules" $V_{m_0}$ $(0 \leq m_0 \leq l-1)$
  3) there is a non trivial braiding on the $V$'s
- Work with $Fun_q(G) = A(G_q)$. as
  1) ~ quantum principal fibre bundles on $SL(2)$ with finite fibres
  2) ~ subcomodules of the regular one

  get:

  $W_{m_1}$ = standard spin $m/2$ of $SL(2)$

  $V_{m_0}$ = exhotic comodules mod $l$
- Study
  - tensor products and decomp.s
  - braiding (induced by R).
- Outlook.

# 1- NOTATIONS AND KNOWN FACTS

Def. $A(sl_q(2))$ = free unital $\mathbb{C}$-algebra with generators $a, b, c, d$

- commutation relations ....
- determinant relation ...

Hopf-algebra structure given by...

Look at the "regular" comodule

$$A \xrightarrow{\Delta} A \otimes A$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \hat{\otimes} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\Rightarrow \mathbb{C}_q[a, c]$ is a left sub comodule algebra

= q-polynomials, graded by the degree

$\simeq \bigoplus_m Y_m$ ; $Y_m \simeq$ space of q-poly of degree $m$

But for $q = \lambda = e^{2\pi i/\ell}$ special facts occour: e.g.

$$\Delta(a^\ell) = (\Delta a)^\ell = (a\otimes a + b\otimes c)^\ell = \sum_r \binom{\ell}{r}_p a^{\ell-r} b^r \otimes a^{\ell-r} c^r$$

same for $b^e, c^e, d^e$ and

$1 = (ad - qbc)^e = a^e d^e - b^e c^e$

So: At $q = \lambda$ ($e$ odd)

1) $A(SL(2)) \longrightarrow A(SL_\lambda(2))$ is a central subalgebra $\simeq$ Hopf alg. of $SL(2)$.

2) $A(SL_\lambda(2))$ is a fintely gerated (actually free) module of $rk = e^3$ over $A(SL(2))$ (i.e. a trivial sheaf, over $SL(2)$).

## 2 - QUANTUM COVERINGS OF $SL(2)$

One can make

$$A(SL(2)) \longrightarrow A(SL_\lambda(2))$$

into a "p.f.b" by looking at

$$A(F) = A(SL_\lambda(2)) / \langle a^e = d^e = 1, \; b^e = c^e = 0 \rangle$$

indeed we have

$$\begin{tikzcd} A(se_\lambda(2)) \arrow[r, "\Delta"] \arrow[dr, "\varphi"'] & A\ldots \otimes A\ldots \arrow[d, "(\pi_F, id)"] \\ & A(F) \otimes A\ldots \end{tikzcd}$$

and $\varphi\big|_{A(se(2))} = 1 \otimes A(se(2))$.

Properties of $A(F)$.

3-. QUANTUM COVERINGS of $SO(3,\mathbb{C})$

$$A(SO(3,\mathbb{C})) \simeq A^{ev}(se(2,\mathbb{C})) \hookrightarrow A(se_\lambda(2))$$

is coinvariant w.r.t.

$$A(\hat{F}) \simeq A(se_\lambda(2)) / \langle a^{2\ell} = d^{2\ell} = 1,\ b^\ell = c^\ell = 0 \rangle$$

Properties of $A(\hat{F})$

of 'functions on finite quantum subgroups' $F$ and $\hat{F}$ of $SL_q(2)$, needed in the sequel.

The Hopf algebra $A(F)$ is defined as the quotient Hopf algebra of $A(SL_q(2))$ modulo the ideal generated by the relations

$$a^\ell = 1 = d^\ell, \quad b^\ell = 0 = c^\ell. \tag{2}$$

Let $\pi_F$ denote the canonical projection, and $\bar{t} := \pi_F(t)$.

We give now some information on $F$ (see [7], [8] for the case $\ell = 3$).

**Proposition 2.1** *$A(F)$ satisfies the following properties:*
*i) as a complex vector space $A(F)$ is $\ell^3$-dimensional and its basis can be chosen as $\bar{a}^p \bar{b}^r \bar{c}^s$, where $p, r, s \in \{0, 1, \ldots, \ell-1\}$.*
*ii) $A(F)$ has a faithful representation $\varrho$*

$$\varrho(\bar{a}) = \mathbf{J} \otimes \mathbf{1}_\ell \otimes \mathbf{1}_\ell\,, \quad \varrho(\bar{b}) = \mathbf{Q} \otimes \mathbf{N} \otimes \mathbf{1}_\ell\,, \quad \varrho(\bar{c}) = \mathbf{Q} \otimes \mathbf{1}_\ell \otimes \mathbf{N}\,, \tag{3}$$

*where for $i, j \in \{1, 2, \ldots, \ell\}$*

$$\mathbf{J}_{i,j} = \begin{cases} 1 & \text{if } i = j+1 \bmod \ell \\ 0 & \text{otherwise} \end{cases}, \quad \mathbf{Q}_{i,j} = \begin{cases} q^{-i} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}, \quad \mathbf{N}_{i,j} = \begin{cases} 1 & \text{if } i = j+1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

*iii) $F$ has the 'reduced' quantum plane as a quantum homogeneous space, i.e. the algebra generated by $x$ and $y$ modulo the ideal generated by the relations $xy = qyx$, $x^\ell = 1$ and $y^\ell = 1$ (isomorphic to $\mathrm{Mat}(\ell, \mathbf{C})$) is a $A(F)$-comodule algebra.*
*iv) $F$ has a classical subgroup, defined as the group of characters of $A(F)$, which is easily seen to be isomorphic to $\mathbf{Z}_\ell$. Namely, for $i \in \{1, 2, \ldots, \ell\}$ we have $\chi_i$ defined by their action on the generators as $\chi_i(\bar{a}) = q^i$, $\chi_i(\bar{b}) = 0$, $\chi_i(\bar{c}) = 0$ and $\chi_i(\bar{d}) = q^{-i}$. The Hopf algebra $A(\mathbf{Z}_\ell)$ appears as a quotient of $A(F)$ by the ideal generated by $\bar{b}$, $\bar{c}$ (which is also the intersection of the kernels of the characters).*

Quite similarly, we define $A(\hat{F})$ as the $2\ell^3$ dimensional quotient of $A(SL_q(2))$ modulo the relations

$$a^{2\ell} = 1 = d^{2\ell}, \quad b^\ell = 0 = c^\ell. \tag{5}$$

Notice that the classical subgroup of $\hat{F}$ is the cyclic group $\mathbf{Z}_{2\ell}$. This group 'combines' the cyclic subgroup $\mathbf{Z}_\ell$ of $F$ with $\mathbf{Z}_2$ appearing in the classical spin cover. In fact one has the exact sequence of groups

$$0 \longrightarrow \mathbf{Z}_2 \longrightarrow \mathbf{Z}_{2\ell} \longrightarrow \mathbf{Z}_\ell \longrightarrow 0 \tag{6}$$

which extends $\mathbf{Z}_\ell$ by the kernel $\mathbf{Z}_2$ of the classical spin cover. [redacted]

## 2.3 Quantum group covering of $SL(2)$

The Hopf subalgebra of $A(SL_q(2))$ generated by the $\ell^{th}$ powers

$$\alpha = a^l,\ \beta = b^l,\ \gamma = c^l,\ \delta = d^l$$

is isomorphic to the (commutative) Hopf algebra

$$A(SL(2)) = \mathbb{C}[\alpha, \beta, \gamma, \delta] \,/\langle \alpha\delta - \beta\gamma - 1\rangle$$

with the restricted coproduct, counit and coinverse. It is just the subalgebra of coinvariants of the coaction of $A(F)$ (as a quotient Hopf algebra). It is known [2] (see also [7], for the case $\ell = 3$) that

A(SO(3))

**Proposition 2.2** *The sequence of algebras*

$$A(SL(2)) \longrightarrow A(SL_q(2)) \xrightarrow{\pi_F} A(F) \tag{7}$$

*is:*
*i) a (right, faithfully flat) Hopf-Galois extension of $A(SL(2)$ by $A(F)$ (quantum principal fibre bundle)*
*ii) a principal homogeneous Hopf-Galois extension (i.e., $A(F)$ is a quotient of the Hopf algebra $A(SL_q(2))$ by a Hopf ideal and $\pi_F$ is the canonical surjection)*
*iii) strictly exact (quantum quotient group)*

(Notice that iii $\Rightarrow$ ii $\Rightarrow$ i, see [29], [26], [28] for the relevant definitions).

Therefore (7) is a good candidate for a quantum covering of the spin group. To be fully worthy of this name, it would be nice if it were nontrivial. It can be seen that it is not totally trivial in the sense that $A(SL_q(2))$ is not isomorphic to $A(SL(2)) \otimes A(F)$. Another accepted notion to substitute the triviality for quantum principal bundles is however that of cleftness (or crossed products), c.f. e.g. [7]. It is not yet known if (7) is cleft and it seems to be a tough problem, which is not tractable by the usual means (e.g. the theory of quantum characteristic classes). A weaker result affirms that $A(SL_q(2))$ as a module over $A(SL(2))$, which is finitely generated and projective (c.f. [11]), is actually free [9]. (The associated coherent sheaf of rank $l^3$ is free and the corresponding vector bundle $F$ over $SL(2)$ turns out to be trivial). Moreover a set of $l^3$ generators can be chosen as [9]

$$a^m b^n c^{s'}, \qquad b^n c^{s''} d^r,$$

with the integers $m, n, r, s', s''$ in the range $m \in \{1, \dots, \ell-1\}$, $n, r \in \{0, \dots, \ell-1\}$, $s' \in \{m, \dots, \ell-1\}$ and $s'' \in \{0, \dots, \ell-r-1\}$.

We expect that the quantum principal bundle (7) is actually non cleft and propose to employ it for a quantum spin covering in the next section.

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## 2.4 Quantum group covering of $SO(3,\mathbb{C})$

The (complex) group $SL(2)$ is isomorphic to the spin group $Spin(3,\mathbb{C})$ and thus provides a twofold covering of the (complex) rotation group $SO(3,\mathbb{C})$. This classical spin covering can be combined with the covering (7) as follows. The Hopf algebra $A(SO(3,\mathbb{C}))$ can be identified with the subalgebra of even polynomials $A^+(SL(2))$ in $A(SL(2))$. It can be seen that $A^+(SL(2))$ coincides with the coinvariants of the coaction of Hopf algebra $A(\hat{F})$. Let $\pi_{\hat{F}}$ denote the canonical projection and $\hat{t} := \pi_{\hat{F}}(t)$. In analogy with the proof of Prop. 2.2 it can be shown that

**Proposition 2.3** *The sequence of Hopf algebras*

$$A(SO(3,\mathbb{C}))\longrightarrow A(SL_q(2)) \xrightarrow{\pi_{\hat{F}}} A(\hat{F}) \ , \tag{8}$$

*possesses the same nice properties i)-iii) as the sequence (7).*

In particular (8) is a quantum principal bundle and referring to our remarks at the end of the previous subsection, we mention that it is very likely non cleft. In fact the relevant question about (8) is if it is reducible to the subgroup $\mathbb{Z}_2$. We expect that it is not the case (consistent with our conjecture about the noncleftness of (7)) and thus propose

**Definition** For any odd $\ell \geq 3$, with $q = e^{\frac{2\pi i}{\ell}}$, we call the principal fibre bundle (8) quantum spin covering of the complex rotation group.

# 3 Irreducible corepresentations at $q=\lambda$, $\lambda^\ell=1$

There are two natural series of corepresentations of $A(SL_q(2))$. The first one comes by restricting the coproduct to $W_n = \mathbb{C}\{\alpha^n, \alpha^{n-1}\gamma, \ldots, \gamma^n\}$, i.e. the span of monomials of degree $n$ in $\alpha = a^\ell$ and $\gamma = c^\ell$. The corepresentation $W_n$ is a 'push forward' of the usual $n+1$ dimensional (spin $n/2$) corepresentations of $A(SL(2))$ and thus it is obviously irreducible for all $n \in \mathbb{N}$. The second one comes by restricting the coproduct to $Y_m = \mathbb{C}\{a^m, a^{m-1}c, \ldots, c^m\}$, i.e. the span of monomials of degree $m$ in $a, c$. More explicitly, $W_n$ $Y_m$

$$\begin{aligned}\Delta a^{m-h}c^h &= \textstyle\sum_{r=1}^{m-h}\sum_{s=1}^{h} q^{-r(h-s)}\binom{m-h}{r}_{q^{-2}}\binom{h}{s}_{q^{-2}} a^{m-h-r}b^r c^{h-s}d^s \otimes a^{m-(r+s)}c^{r+s} \\ &= \textstyle\sum_{k=0}^{m}\left(\sum_{r+s=k} q^{-r(h-s)}\binom{m-h}{r}_{q^{-2}}\binom{h}{s}_{q^{-2}} a^{m-h-r}b^r c^{h-s}d^s\right) \otimes a^{m-k}c^k \ , \end{aligned} \tag{9}$$

where

$$\binom{k}{j}_p = \frac{(k)!_p}{(k-j)!_p (j)!_p} \qquad (k)!_p = (k)_p(k-1)_p \ldots (2)_p \ \text{ and } \ (k)_p = 1 + p + \ldots + p^{k-1} \ .$$

It is indecomposable but not irreducible in general. As we shall see, for $m \in \{0, 1, \ldots, \ell{-}1\}$ $Y_m$ is indeed irreducible and we shall denote it $V_m$. Also, for $m = n\ell - 1$, $n \in \mathbb{Z}_+$, it is irreducible but in fact equivalent to $W_{n-1} \otimes V_{\ell-1}$. More generally the corepresentations of the form $W_n \otimes V_m$, with $n \in \mathbb{N}$, $m \in \{0, 1, \ldots, \ell-1\}$ are all irreducible as can be inferred from [11], [10], establishing the duality with a version [12] of divided powers algebra [20], [21] of which the classification of irreducible representations is known [20]. Although straightforward, here we provide a direct computational proof.

**Proposition 3.1** *Set $m = m_0 + \ell m_1$, with $0 \leq m_0 \leq \ell - 1$, $m_1 \geq 0$.*
*a) For $m_1 = 0$ the comodule $V_{m_0} := Y_{m_0}$ is irreducible.*
*a′) For every $m_1 > 0$, the comodule $Y_{\ell-1+\ell m_1}$ (i.e. when $m_0 = \ell - 1$) is irreducible as well and it is isomorphic to $W_{m_1} \otimes V_{\ell-1}$.*
*b) when $0 \leq m_0 \leq \ell - 2$ and $m_1 \geq 1$, $Y_{m_0+\ell m_1}$ has a (maximal) subcomodule isomorphic to $W_{m_1} \otimes V_{m_0}$. The quotient comodule is irreducible and isomorphic to $W_{m_1-1} \otimes V_{\ell-2-m_0}$.*

**Corollary.** *The corepresentations $\boxed{W_n \otimes V_m}$ are irreducible for all $n \in \mathbb{N}$ and $m \in \{0, 1, \ldots, \ell-1\}$.*

*Proof of Proposition (3.1).* The classical argument working for $q = 1$ can be directly extended when $q$ is considered as an indeterminate and runs as follows. Given a corepresentation $\rho$ of a Hopf algebra $A$ on a comodule $U$, let $\rho_i^j$ be a matrix of elements of $A$ such that

$$u_i \mapsto \rho_i^j \otimes u_j,$$

with respect to a basis $u_i$ ($i = 1, \ldots, n$) of $U$. There exist a coinvariant subcomodule $U' \subset U$ (with $\dim U' = k$, say) iff up to a conjugation by an invertible matrix $Z$ with elements in $\mathbb{C}[q, q^{-1}]$ the matrix $\rho$ takes a lower echelon form, i.e. iff

$$\begin{pmatrix} \tau_1 & 0 \\ \tau_3 & \tau_4 \end{pmatrix} \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \begin{pmatrix} \rho_1 & \rho_2 \\ \rho_3 & \rho_4 \end{pmatrix}$$

where $U'$ is the span of the first $k$ elements of the transformed basis and the block decomposition is given by the splitting $U = U' \oplus U/U'$. In particular this requires that

$$\tau_1 \,( z_1 \quad z_2 ) = ( z_1\rho_1 + z_2\rho_3 \quad z_1\rho_2 + z_2\rho_4 ) .$$

Notice that the $k \times n$ matrix $( z_1 \quad z_2 )$ has rank $k$. Let $M$ be an invertible $k \times k$ submatrix. We can write

$$\tau_1 M = S, \qquad \tau_1 M' = S' ,$$

where $S$ is the submatrix of the r.h.s. above corresponding to the columns of $M$ in $( z_1 \quad z_2 )$ and prime denotes the submatrices with the complementary columns. Substituting, we get $k \times (n-k)$ linear relations over $\mathbb{C}[q, q^{-1}]$ among the elements $\rho_i^j$

$$S' = S M^{-1} M' .$$

Now let's have a closer look at the comodules $Y_m$. The matrix elements of (9) are linear combinations of monomials of degree $m$ in the generators $a, b, c, d$. For generic $q$, all



$$\rho_h^k = \sum_{r+s=k} \ldots \binom{m-h}{r}_{q^{-2}} \binom{h}{s}_{q^{-2}} a^{m-h-r} b^r c^{h-s} d^s$$

$Y_m$

$(m+3)(m+2)(m+1)/6$ of them appear in the sum on the r.h.s. of (9) and every monomial appears exactly in a single matrix element i.e. two different matrix elements contain different monomials. Therefore they are all linearly independent and, by the argument above, the $Y_m$ are irreducible.

Recall that when $q = \lambda$ is a $\ell^{th}$ root of unity, $\lambda^\ell = 1$, the subalgebra generated by $\alpha = a^\ell$, $\beta = b^\ell$, $\gamma = c^\ell$, $\delta = d^\ell$ is central and isomorphic to the classical Hopf algebra $A(SL(2))$. Now several $q^{-2}$-binomial coefficients actually vanish when evaluated at $\lambda$. A simple way to control this is to use the fact that the coproduct is an algebra homomorphism;

$$\Delta a^m = \Delta \alpha^{m_1} \Delta a^{m_0} \ ,$$

$m = \ell m_1 + m_0$

where $m = m_0 + \ell m_1$. Hence

$$\sum_{r=0}^{m} \binom{m}{r}_{\lambda^{-2}} a^{m-r} b^r \otimes a^{m-r} c^r =$$

$$\sum_{i=0}^{m_1} \sum_{j=0}^{m_0} \binom{m_0}{j}_{\lambda^{-2}} \binom{m_1}{i}_1 \alpha^{m_1-i} \beta^i a^{m_0-j} b^j \otimes \alpha^{m_1-i} \gamma^i a^{m_0-j} c^j,$$

giving the factorization formula (c.f. [20])

$$\binom{m}{r}_{\lambda^{-2}} = \binom{m_0}{r_0}_{\lambda^{-2}} \binom{m_1}{r_1}, \tag{10}$$

where $m = m_0 + \ell m_1$, $r = r_0 + \ell r_1$, $0 \le m_0, r_0 \le \ell - 1$, $m_1, r_1 \ge 0$ and last factor on the r.h.s. is just the ordinary binomial coefficient. In particular all the binomial coefficients $\binom{m}{r}_{\lambda^{-2}}$ with $r_0 > m_0$ vanish.

When $0 \le m = m_0 \le \ell - 1$, this can not occur and the standard argument above yields the point a). For larger $m$'s however the comodule $Y_m$ is no more irreducible. Using again the homomorphism property of the coproduct, we directly compute

$$\Delta(a^{m-h} c^h) \ = \ \Delta(\alpha^{(m-h)_1} \gamma^{h_1}) \times$$

$$\sum_{j=0}^{(m-h)_0} \sum_{t=0}^{h_0} \lambda^{\,j(h_0-t)} \binom{(m-h)_0}{j}_{\lambda^{-2}} \binom{h_0}{t}_{\lambda^{-2}} a^{(m-h)_0-j} b^j c^{h_0-t} d^t \otimes a^{(m-h)_0+h_0-(j+t)} c^{j+t} .$$

Notice that

$$m - h = \begin{cases} \ell(m_1 - h_1) + (m_0 - h_0) & \text{if } \ 0 \le h_0 \le m_0 \\ \ell(m_1 - h_1 - 1) + (\ell + m_0 - h_0) & \text{if } \ m_0 + 1 \le h_0 \le \ell - 1 \end{cases}$$

and the formula above for $h_0 \le m_0$ reads

$$\Delta(a^{m-h}c^{h}) = \Delta(\alpha^{m_1-h_1}\gamma^{h_1})\times \tag{11}$$

$$\sum_{j=0}^{m_0-h_0}\sum_{t=0}^{h_0}\lambda^{-j(h_0-t)}\binom{m_0-h_0}{j}_{\lambda^{-2}}\binom{h_0}{t}_{\lambda^{-2}}a^{m_0-h_0-j}b^{j}c^{h_0-t}d^{t}\otimes a^{m_0-(j+t)}c^{j+t}.$$

This sum contains only monomials in $a, c$ of degree $m_0$. So $W_{m_1}\otimes V_{m_0}$ is an irreducible subcomodule. If $m_0 = \ell - 1$ this is isomorphic to the whole of $Y_{\ell-1+\ell m_1}$. This proves the point a') and the first statement of the point b). To complete the proof of b) notice that for $m_0 + 1 \le h_0 \le \ell - 1$ we have

$$\Delta(a^{m-h}c^{h}) = \Delta(\alpha^{m_1-h_1-1}\gamma^{h_1})\times$$

$$\sum_{j=0}^{\ell+m_0-h_0}\sum_{t=0}^{h_0}\lambda^{-j(h_0-t)}\binom{\ell+m_0-h_0}{j}_{\lambda^{-2}}\binom{h_0}{t}_{\lambda^{-2}}a^{\ell+m_0-h_0-j}b^{j}c^{h_0-t}d^{t}\otimes a^{\ell+m_0-(j+t)}c^{j+t}.$$

Restricting the sum to $j+t \le m_0$ we can factor $a^{\ell} = \alpha$, while restricting to $j+t \ge \ell$ we can factor $c^{\ell} = \gamma$, thus compensating the $-1$ occurring in the exponent of the classical part of the coproduct and leaving in these two partial sums only monomials of degree $m_0$ in $a, c$. This cannot be done in the partial sum for $m_0 + 1 \le j + t \le l - 1$, which gives

$$\sum_{k=m_0}^{l-1}\sum_{s=0}^{k}\ldots a^{l+m_0-h_0-k+s}b^{k-s}c^{h_0-s}d^{s}\otimes a^{l+m_0-k}c^{k} =$$

$$\sum_{k'=0}^{l-m_0-2}\sum_{s}^{m_0+1+k'}\ldots a^{m_0'-h_0'-k'+s}b^{m_0+1+k'-s}c^{m_0+1+h_0'-t}d^{t}\otimes (ac)^{m_0+1}a^{m_0'-k'}c^{k'} =$$

$$\lambda^{n}[(bc)^{m_0+1}\otimes(ac)^{m_0+1}]\Delta a^{m_0'-h_0'}c^{h_0'},$$

where $m_0' = l - m_0 - 2$, $h_0' = h_0 + m_0 + 1$. This expression contains monomials of degree larger than $m_0$. Thus the quotient $Y_m/W_{m_1}\otimes V_{m_0}$ is isomorphic to $W_{m_1-1}\otimes V_{\ell-2-m_0}$ which completes the proof. □

We remark that the proof for generic $q$ is just the q-analogue of what happens in the classical case. It is enough to notice that replacing the ordinary binomial coefficients the q-analogue one gets (9) up to some nonvanishing factors. This is why the corepresentation theory for generic $q$ is the same as the classical one. In particular the argument above yields that the Clebsh-Gordan decomposition holds as in the classical case

$$Y_m \otimes Y_{m'} = Y_{m+m'} \oplus Y_{m+m'-2} \oplus \ldots \oplus Y_{|m-m'|}\,.$$

In the case of $q$ being $\ell$-th root of unity it follows from Prop. 3.1 that there is always at least one irreducible corepresentation of arbitrary dimension $D$, and there are more (up to $\ell$) of them, according to how many integers in $\{1, \ldots, \ell\}$ divide $D$.

The decomposition rules of the tensor products $(W_n\otimes V_m)\otimes(W_{n'}\otimes V_{m'})$ into irreducible corepresentations follow from the usual Clebsh-Gordan decomposition of $W_n\otimes W_{n'}$ and the decomposition of $V_m\otimes V_{m'}$, which obey a more complicated pattern.

### 3.1 Decomposition of tensor products for $\ell = 3$

The previous discussion can be specified and simplified considerably in the simplest (non-trivial) case $\ell = 3$. We have explicitly the following matrices $\rho$ of three corepresentations $V_0$, $V_1$ and $V_2$ respectively

$$1, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \begin{pmatrix} a^2 & -q^2ab & b^2 \\ ac & ad + q^{-1}bc & bd \\ c^2 & -q^2cd & d^2 \end{pmatrix};$$

as well as the usual form of $W_n$.

According to the corollary of proposition 3.1 we see that there is one trivial (onedimensional, irreducible) corepresentation $V_0 = W_0$. In dimension two there are two (inequivalent) irreducible corepresentations $V_1$ and $W_1$. In dimension three there are also two: $V_2$ and $W_2$. In dimension four $Y_3$ is indecomposable but not irreducible but there are two other irreducible corepresentations $W_3 \otimes V_0 = W_3$ and $W_1 \otimes V_1$. In dimension five there is only one irreducible corepresentation $W_4$ ($Y_4$ is indecomposable but not irreducible). In dimension six there are three irreducible corepresentations: $W_5$, $W_2 \otimes V_1$ and $W_1 \otimes V_2 = V_5$. A general pattern is that in any dimension $D$ there is always at least one irreducible corepresentation, if either 2 or 3 divide $D$ there are two irreducible corepresentations and if 6 divides $D$ there are three irreducible corepresentations.

We give now the decomposition rules of the tensor products.

Clearly,

$$V_0 \otimes V_0 = V_0, \quad V_0 \otimes V_1 = V_1 \text{ and } V_0 \otimes V_2 = V_2 .$$

Next, it can be seen that

$$V_1 \otimes V_1 = V_0 \oplus V_2, \quad V_1 \otimes V_2 = V_1 \oslash W_1 \oslash V_1 \text{ and } V_2 \otimes V_2 = V_0 \oslash V_2 \oslash (W_1 \oplus V_1) \oslash V_0 ,$$

where $\oslash$ in an indecomposable corepresentation indicates that the left summand is a sub-comodule while the right summand is a comodule only after quotienting the left one. Noting that the tensor products in the opposite order decompose equivalently, and recalling the usual decomposition $W_n \otimes W_{n'} = W_{|n-n'|} \oplus W_{|n-n'|+2} \oplus \ldots \oplus W_{|n+n'|}$, these rules permit to find a decomposition of tensor products of any number of $V_m \otimes W_n$, with $n \in \mathbb{N}$ and $m \in \{1, 2, 3\}$.

"An interesting question in this simplest (nontrivial) case of $\ell = 3$ is if there is a possibility to built a fundamental fermion out of three anyons. The decomposition rules permit to verify easily that the corepresentations $(V_1)^{\otimes 3}$ and $(V_2)^{\otimes 3}$ do not contain the fundamental spinor corepresentation $W_1$ as a subcorepresentation and the same is true for the tensor cube of the irreducible corepresentations $(V_m \otimes W_n)$ if $m \in \{1, 2\}$. They do however contain $W_1$ as a quotient (sub)corepresentation. It is also worth to mention that the fundamental spinor $W_1$ subcorepresentation occurs nevertheless in e.g. the reducible but not decomposable representation $Y_3$ and thus also in its third tensor power $(Y_3)^{\otimes 3}$."

| | |
|---|---|
| 1: $V_0 \simeq W_0$ | 4: $V_1 \otimes W_1$, $W_3$ |
| 2: $V_1$, $W_1$ | 5: $W_4$ |
| 3: $V_2$, $W_2$ | 6: $V_2 \otimes W_1$, $V_1 \otimes W_2$, $W_5$ |

# 4 Braiding

For general $q$ the quasitriangular structure on $U_q(sl(2))$ given by the well known universal element $R$ [14] in (a suitable completion of) $U_q(sl(2))^{\otimes 2}$, defines a coquasitriangular structure on $A(SL_q(2))$. Its explicit form on the generators reads (c.f. [17])

$$\mathcal{R}\begin{pmatrix} a\otimes a & a\otimes b & a\otimes c & a\otimes d \\ b\otimes a & b\otimes b & b\otimes c & b\otimes d \\ c\otimes a & c\otimes b & c\otimes c & c\otimes d \\ d\otimes a & d\otimes b & d\otimes c & d\otimes d \end{pmatrix} = \begin{pmatrix} q^{-1/2} & 0 & 0 & q^{1/2} \\ 0 & 0 & q^{-1/2}-q^{3/2} & 0 \\ 0 & 0 & 0 & 0 \\ q^{1/2} & 0 & 0 & q^{-1/2} \end{pmatrix}. \tag{12}$$

This structure provides a highly unusual (nonsymmetric and nondiagonal) braiding of corepresentations $\rho$ and $\rho'$ of $A(SL_q(2))$

$$\Psi(u_i\otimes u'_r) = \sum_{j,s} \mathcal{R}\left({\rho'}_r{}^s \otimes \rho_i{}^j\right) u'_s\otimes u_j \,. \tag{13}$$

In our situation, $q^\ell = 1$, it is not difficult however to verify that the corepresentations $W_n$ for $n$ odd (i.e. with halfinteger spin $n/2$) are fermionic and the corepresentations $W_n$ for $n$ even (i.e. with integer spin $n/2$) are bosonic, i.e. they obey

$$\Psi(w\otimes w') = (-1)^{nn'} w'\otimes w \text{ for } w \in W_n,\ w' \in W_{n'} \,. \tag{14}$$

Thus the exotic coquasitriangular structure $\mathcal{R}$ passes an important consistency test of the agreement with the usual spin-statistics relation in dimensions $d > 3$. The braiding of $V$'s and $W$'s is also quite simple:

$$\Psi(v\otimes w) = (-1)^{mn} w\otimes v \text{ for } v \in V_m,\ w \in W_n \,. \tag{15}$$

Instead the braiding of $V$'s among themselves is highly exotic (even comparing with the anyonic one). We report it in the Subsection 4.1 for the case $\ell = 3$. Clearly the tensor products $V_m\otimes W_n$ carry the combined statistics according to the usual hexagon conditions for $\Psi$ (see eg. [23]).

## 4.1 Braiding in the case $\ell = 3$

$q = \lambda\,,\ \lambda^3 = 1$

As for general $\ell$, the braiding of the corepresentations $W_n$ with $W'_n$ is exactly the classical one, i.e. the trivial twist except when $nn'$ is odd when it is (-) the twist. Also, $W_n$ have the trivial braiding with $V_0$ and with $V_2$ and (-) the twist with $V_1$. The braiding of $V$'s among themselves is as follows:
The braiding of $V_1$ and $V_1$ reads:

$$\Psi\begin{pmatrix} a\otimes a \\ a\otimes c \\ c\otimes a \\ c\otimes c \end{pmatrix} = \begin{pmatrix} q^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & q^{1/2} & 0 \\ 0 & q^{1/2} & 1+q^{-1/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \end{pmatrix}\begin{pmatrix} a\otimes a \\ a\otimes c \\ c\otimes a \\ c\otimes c \end{pmatrix}. \tag{16}$$

(Notice that $\Psi$ has a nonsimple tensor $a\otimes c - qc\otimes a$ as an eigenvector with eigenvalue 1 and the complex span of $a\otimes a, qa\otimes c + c\otimes a, c\otimes c$ as an eigenspace with eigenvalue $q^{-1/2}$). Next, the braiding of $V_1$ and $V_2$ reads:

$$\Psi\begin{pmatrix} a^2\otimes a \\ a^2\otimes c \\ ac\otimes a \\ ac\otimes c \\ c^2\otimes a \\ c^2\otimes c \end{pmatrix} = \begin{pmatrix} q^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 \\ 0 & 1 & 0 & q^2-q & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & q & 0 & 1-q & 0 \\ 0 & 0 & 0 & 0 & 0 & q^2 \end{pmatrix}\begin{pmatrix} a\otimes a^2 \\ a\otimes ac \\ a\otimes c^2 \\ c\otimes a^2 \\ c\otimes ac \\ c\otimes c^2 \end{pmatrix} \tag{17}$$

The opposite braiding of $V_2$ and $V_1$ reads:

$$\Psi\begin{pmatrix} a\otimes a^2 \\ a\otimes ac \\ a\otimes c^2 \\ c\otimes a^2 \\ c\otimes ac \\ c\otimes c^2 \end{pmatrix} = \begin{pmatrix} q^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 \\ 0 & q & q-q^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1-q^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^2 \end{pmatrix}\begin{pmatrix} a^2\otimes a \\ a^2\otimes c \\ ac\otimes a \\ ac\otimes c \\ c^2\otimes a \\ c^2\otimes c \end{pmatrix}. \tag{18}$$

Finally, the braiding of $V_2$ and $V_2$ reads:

$$\Psi\begin{pmatrix} a^2\otimes a^2 \\ a^2\otimes ac \\ a^2\otimes c^2 \\ ac\otimes a^2 \\ ac\otimes ac \\ ac\otimes c^2 \\ c^2\otimes a^2 \\ c^2\otimes ac \\ c^2\otimes c^2 \end{pmatrix} = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 0 & 0 \\ 0 & 1 & 0 & 1-q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1-q^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & q^2 & 0 & q-1 & 0 & -(q-1)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1-q & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^2 \end{pmatrix}\begin{pmatrix} a^2\otimes a^2 \\ a^2\otimes ac \\ a^2\otimes c^2 \\ ac\otimes a^2 \\ ac\otimes ac \\ ac\otimes c^2 \\ c^2\otimes a^2 \\ c^2\otimes ac \\ c^2\otimes c^2 \end{pmatrix}. \tag{19}$$

The resulting braiding of the ireeducible corepresentations $W_n\otimes V_m$ and $W_{n'}\otimes V_{m'}$ can be obtained from the above braidings of $V$'s and $W$'s using the hexagon conditions for braiding.

## 5 Final remarks

We remark in a connection with the point iii) of Prop. 2.1 that Mat$(3,\mathbb{C})$ occurs as a direct summand of Connes' interior algebra $\mathcal{A}$ for the Standard Model [4]. Also, the algebra $M(3,\mathbb{C}) \oplus M(2,\mathbb{C}) \oplus \mathbb{C}$, close to $\mathcal{A}$, coincides with the semisimple part of the algebra $U_q(sl(2))$ at cubic roots of unity, which was extensively studied (cf. [6], [18], [8] and references therein).