Non-Commutative Geometry in a model for Turbulence. 1.

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Any idea for quantum gravity has a 'down to earth' application to hydrodynamics. For, they share the same symmetry group: Diff (M).

'Regularization' of hydrodynamics needed for (i) numerical solution
(ii) conceptual methods such as a formula for entropy.

So NCG could be useful in understanding classical hydrodynamics.

We will work on two-dimensional incompressible inviscid hydrodynamics as a manageable first step.

Results:- (i) formula for entropy
(ii) Shape of vorticity distribution in a "hurricane."

Euler's Equations.

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\, v = -\vec{\nabla} p + \vec{f} \rightsquigarrow \text{set } \vec{f} = 0$$

$$\vec{\nabla}\cdot\vec{v} = 0$$

Eliminate p by taking curls

$$\frac{\partial \vec{\omega}}{\partial t} + \mathcal{L}_v \vec{\omega} = 0$$

$$\vec{\omega} = \nabla \times \vec{v}$$

Analogous to eqn. for a rigid body:

$$\frac{\partial \vec{L}}{\partial t} + \vec{\Omega} \times \vec{L} = 0$$

$$\vec{L} = \mathbb{I}\,\vec{\Omega} \qquad \mathbb{I} = \text{Moment of inertia tensor}$$

Hamiltonian formalism:-

$$\{L_i, L_j\} = \epsilon_{ijk} L_k \rightsquigarrow SO(3)$$

$$H = \frac{1}{2} L_i [\mathbb{I}^{-1}]_{ij} L_j$$

There is a similar Hamiltonian formalism for Euler equations of hydrodynamics. The group of rotations is replaced by volume preserving Diffeomorphisms. The inertia tensor by the diff'l. operator 'Curl'. The Hamiltonian is

$$H = \int \frac{1}{2} \vec{v}^{\,2} d^3x$$

Two-Dimensional Hydrodynamics

Vorticity is a scalar; it determines velocity through

$$v_a(x) = \epsilon_{ab}\, \partial_b \int G(x;y)\, \omega(y)\, d^2y$$

$$-\partial^2 G(x,y) = \delta^2(x-y).$$

$$\frac{\partial \omega}{\partial t} + v_a \partial_a \omega = 0$$

Infinite number of conserved quantities

$$Q_k = \int \omega^k(x)\, d^2x \quad ; \ k = 1, 2, 3 \ldots$$

$Q_1 \equiv$ total vorticity; $Q_2 =$ "Enstrophy" etc.

System is not integrable: the submanifold with $Q_k =$ const. is still infinite dimensional.

What is the volume of this phase space?

Recall from Boltzmann

Entropy $\equiv$ log(volume) of phase space with fixed value for conserved quantities.

Need to find volume of infinite dimensional submanifold.

Symplectic Diffeomorphisms

Area preserving $\equiv$ symplectic.

let (M, ρ) be a 2-manifold with area form ρ

let $f_{1,2}: M \to R$

The Poisson br: $\{f_1, f_2\} = \rho^{-1}(df, dg)$

defines a Lie algebra: analogous to the rotational Lie algebra for a rigid body. Let $\omega_f = \int f(x)\, \omega(x)\, d^2x$

[Choose co-ordinates so that $\rho = 1$]

$$\{\omega_{f_1}, \omega_{f_2}\} = \omega_{\{f_1, f_2\}}$$

$$H = \frac{1}{2}\int \omega(x)\,\omega(y)\,G(x,y)\,d^2x\,d^2y \equiv \frac{1}{2}\int v^2(x)\,d^2x$$

$$\Rightarrow \quad \frac{\partial \omega}{\partial t} + \mathcal{L}_v \omega = 0.$$

$Q_k = \int \omega^k d^2x$ is in the center just as $L^2 = L_1^2 + L_2^2 + L_3^2$ commutes with L_i.

Regularization of symplectomorphisms $\equiv$ unitary transformations on a finite-dimensional Hilbert space.

Fairlie & Zachos 1988 PLB

Fourier Analysis

Let $M = S^1 \times S^1$ ie
$$\omega(x_1+L_1, x_2) = \omega(x_1, x_2)$$
$$\omega(x_1, x_2+L_2) = \omega(x_1, x_2)$$

$$\omega = \sum \omega_{m_1 m_2} \, e^{2\pi i\left[m_1 \frac{x_1}{L_1} + m_2 \frac{x_2}{L_2}\right]}$$

$$H = \frac{1}{2} \sum_{\substack{m \in \mathbb{Z}^2 \\ m \neq (0,0)}} \frac{|\omega_m|^2}{m_1^2 + m_2^2} \qquad m, n \in \mathbb{Z}^2.$$

$$\{\omega_m, \omega_n\} = i\, m \times n \; \omega_{m+n}$$

Exactly analogous to
$$H = \frac{1}{2}\left[\frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}\right]$$
$$\{L_i, L_j\} = \epsilon_{ijk} L_k$$
for the Rigid body.

Not good to "cut-off" sum at $|m| < N$:
coupling of large and small momenta.
Instead, replace by addition modulo N:-

$$\{\omega_m, \omega_n\} = i \frac{N}{2\pi} \sin \frac{2\pi\, m \times n}{N} \; \omega_{m+n|N}$$

$$H = \frac{1}{2} \sum_{m_1, m_2 = 0}^{N-1} \left[\frac{|\omega_m|^2}{\sin^2 \frac{2\pi m_1}{N} + \sin^2 \frac{2\pi m_2}{N}}\right] \left(\frac{N}{2\pi}\right)^2$$

..satisfies Jacobi identity! $\equiv$ $\underline{U(N)}$
Regularized hydrodynamics"

Non-Commutative Torus

In essence our regularization replaces $S^1 \times S^1$ by a non-commutative Torus.

$$U_1 U_2 = e^{i\theta} U_2 U_1 \; ; \quad \theta = \frac{2\pi}{N}$$

$$U_1^N = U_2^N = 1.$$

$f : S^1 \times S^1 \to R$ $\qquad f = \sum f_m \, e^{i\left[m_1 x_1 \frac{2\pi}{L} + m_2 x_2 \frac{2\pi}{L}\right]}$

$$\hat{f} = \sum f_m \; U(m)$$

$$U(m) = U_1^{m_1} U_2^{m_2} e^{\pi i m_1 m_2}$$

$$U^+(m)\, U(m) = 1.$$

Replacing f by the operator $\hat{f}$ is a 'deformation' ['quantization'] with $\frac{1}{N}$ playing the role of $\hbar$. But for us this is merely a regularization: we will always be working on classical hydrodynamics.

$$U_1 = \begin{pmatrix} e^{i\theta} & & & \\ & e^{2i\theta} & & 0 \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix}$$

$$\hat{\omega} = \sum_{m_1, m_2 = 0}^{N-1} \omega_m \; U(m) \qquad U_2 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & & & \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

$$\{\omega_{ab}, \omega_{cd}\} = \delta_{bc}\, \omega_{ad} - \delta_{da}\, \omega_{cb}$$

$a, b = 1, \dots N.$

Thus $Q_k = \frac{1}{N} \operatorname{tr} \hat{\omega}^k$ have zero P.B. with ω_{ab}

$$\frac{1}{N} \operatorname{tr} \hat{\omega}^k \longrightarrow \int \omega^k(x) \frac{d^2x}{\text{Area}} \quad \text{as } N \to \infty$$

Regularized Hydrodynamics

$$H = \frac{1}{2} \omega_{ab} \omega_{cd} I_{abcd}$$

I_{abcd} is a complicated tensor.

$$\frac{d\hat{\omega}_{ab}}{dt} = i[X, \omega]$$

$$X_{ab} = I_{abcd}\, \omega_{cd}.$$

Thus $Q_k = \frac{1}{N} \operatorname{tr} \hat{\omega}^k$ are conserved.

It is possible to use this system of ope to solve hydrodynamics numerically.

[F. Dowker]

Advantage: No 'leakage' of energy to high frequency modes.

Geometric meaning: Arnold has shown that Euler eqns. describe geodesics on SDiff(M). The above are geodesics on U(N) w.r.t. a left invariant metric. Not integrable. In fact sectional curvature is negative ⟹ chaotic for N large enough.

Entropy in Hydrodynamics

We are usually not interested in details of a flow. "Averaging out" high frequency modes sufficient. We could replace hamiltonian mechanics by a statistical description.

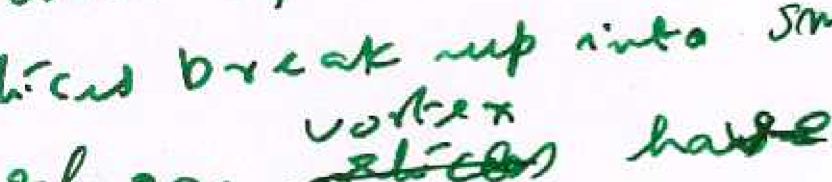
Hamiltonian $\longrightarrow$ Free energy

$$= \text{Energy} - \underbrace{T}_{\text{constant}}\ \text{entropy}$$

Entropy = log(volume of phase space with Q_k = constant).

As time $\rightarrow$ infinity we should expect the system to settle into a state of large entropy.

<u>observed fact</u>:- Reverse cascade in 2D Hydro.
Small vortices merge to form big vortices.



$\left\{ \begin{array}{l} \text{Conserving} \\ \int \omega^k d^2x \text{ for all } k. \end{array} \right.$

[Opposite of behavior in 3D Hydro where vortices break up into smaller ones.]

Does a large ~~vortices~~ vortex have more entropy than a pair of small vortices.

Volume of phase-space.

The set of all hermitian $N\times N$ matrices is our regularized phase space.

The invariants $Q_k = \frac{1}{N}\operatorname{tr} \omega^k$ $k=1,2,\cdots N$

are determined by initial conditions.

[Also, time evolution is a unitary transfn.]

What is the volume of phase space for fixed Q_k? $\equiv$ What is the volume of the set of all hermitian matrices with a given spectrum?

Mehta, Random Matrices: –

$$\text{Volume} = [\text{const.}] \prod_{k<l} (\lambda_k - \lambda_l)^2.$$

$$\text{Entropy} = \frac{1}{N^2} \log \text{volume}$$

$$= \frac{1}{N^2} 2 \sum_{k<l} \ln |\lambda_k - \lambda_l| \; := \mathcal{S}_N$$

$$\rho(\lambda) = \frac{1}{N} \sum_k \delta(\lambda - \lambda_k)$$

$$\mathcal{S}_1 = \int_{\lambda\neq\lambda'} \rho(\lambda)\rho(\lambda') \ln|\lambda-\lambda'| \, d\lambda \, d\lambda'$$

As $N\to\infty$

$$\rho(\lambda) \to \int \delta(\lambda - w(x)) \frac{d^2x}{\text{Area}} + \mathcal{S} \to \int$$

The Removal of Regularization

$$\rho(\lambda) = \frac{1}{N}\sum_{K} \delta(\lambda - \lambda_K)\ ;\quad \int \rho(\lambda)\, d\lambda = 1.$$

$$\to \int \delta(\lambda - w(x))\, \frac{d^2x}{\text{Area}}$$

$\rho(\lambda)\,d\lambda$ is the perimeter of the curve on which $w(x) = \lambda$.

$$S = \beta \int \rho(\lambda)\rho(\lambda') \ln|\lambda - \lambda'|\, d\lambda\, d\lambda'$$

Thus

$$S = \int \ln|w(x) - w(y)|\ \frac{d^2x}{\text{Area}}\ \frac{d^2y}{\text{Area}}$$

Remarkably simple formula.

If $w(x) = \text{const}$, $\quad S \to -\infty.$

"ordered state".

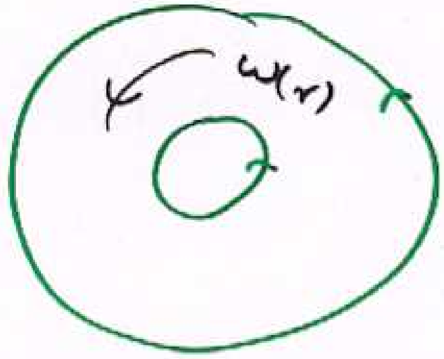
If we combine vortices $\overset{\omega_1}{\circ}$ $\overset{\omega_2}{\circ}$ into a big one ω entropy grows!

$$W_{in} = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right] \qquad W_{fin} = \begin{pmatrix} G \end{pmatrix}$$

The volume of phase space bigger if matrix not required to be block-diagonal.

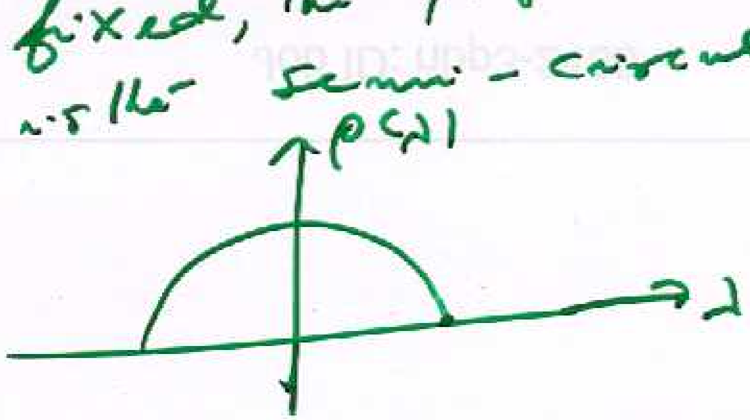
Profile of a Vortex.

Consider axially symmetric flow in between two circles of radii R_1; R_2



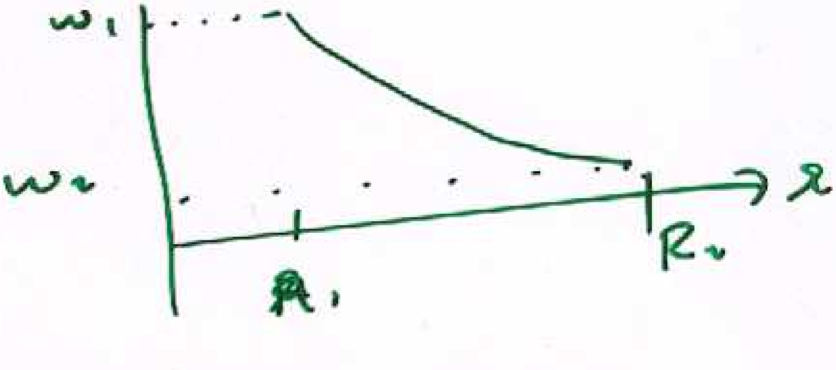
Any profile $\omega(r)$ is a static solution of Euler's equations. Which one has the most entropy?

If the mean vorticity and the "enstrophy" are held fixed, the profile with largest entropy is the semi-circular distribution



$$\omega(r)\, 2\pi r\, dr = \rho(\omega(r))$$

We can solve for parametric solution given in our paper.



3 Dimensional Hydrodynamics

Incompressible $\Rightarrow$ $\mathcal{L}_v \rho = 0 \Rightarrow \vec{\nabla}\cdot\vec{v} = 0$

Wd- symplectic. What are finite dimensional deformation of the Lie algebra of incompressible vector fields!

since

1. Hopf algebra related to the foliation

2. If $\vec{v} \neq 0$ everywhere, we have a foliation. The "basic functions" $\mathcal{L}_v f = 0$ form a Poisson-algebra.

$$\{f, g\}_v = i_v \rho(df, dg)$$

Thus each incompressible vector field defines a Poisson algebra.

3. "Clebsh variables"

$$\vec{v} = \vec{\nabla} p \times \vec{\nabla} q \Rightarrow \vec{\nabla}\cdot\vec{v} = 0$$

$$\{p(x), q(y)\} = \delta^3(x-y)$$

$$\{p(x), p(y)\} = \{q(x), q(y)\} = 0.$$

$$H = \frac{1}{2}\int (\nabla p \times \nabla q)^2 d^3x.$$

Regularization by deformation?