Quantum biHamiltonian Systems

and

Alternative $*$-products

G. MARMO

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Ideology:

DIRAC :

"Classical mechanics must be a limiting case of quantum mechanics. We should thus expect to find that important concepts in classical mechanics correspond to important concepts in quantum mechanics and, from an understanding of the general nature of the analogy between classical and quantum mechanics, we may hope to get laws and theorems in quantum mechanics appearing as simple generalizations of well known results in classical mechanics ..."

Natural questions

Quantum counterpart of biHamiltonian descriptions available for completely integrable systems at the classical level

Quantum version of reduction and unfolding procedures

Constraints in a purely quantum framework

Starting from

$$\ddot{x}_j = f_j(x, \dot{x}) \quad \text{on } Q$$

solve for $\{\ ,\ \}$, H the

following equation

$$\dot{x}_j = \{H, x_j\}$$

$$\ddot{x}_j = \{H, \{H, x_j\}\}$$

Feynman

$$\{x_j, x_k\} = 0$$

Motivation

$$\ddot{x}_j = -\omega^2 x_j + k\, \epsilon_{jk} \dot{x}_k \qquad k = \frac{e}{c} B$$

Alternative descriptions

$$\{ p_j , x_k \} = \delta_{jk} \qquad \{ x_j , x_k \} = 0 = \{ p_j , p_k \}$$

$$H = \frac{p^2}{2} + [\omega^2 + (\tfrac{k}{2})^2]\, x^2 - \frac{k}{2} J \qquad J = \epsilon_{jk} x_j p_k$$

$$\{ p_j , x_k \} = \delta_{jk} \qquad \{ x_j , x_k \} = \frac{k}{\omega^2} \epsilon_{jk} \qquad \{ p_j , p_k \} = 0$$

$$H = \frac{p^2}{2} + \frac{\omega^2}{2} x^2$$

$$\{ u_j , x_k \} = \delta_{jk} \qquad \{ u_j , u_e \} = \epsilon_{je}\, k \qquad \{ x_j , x_k \} = 0$$

$$H = \frac{u^2}{2} + \frac{\omega^2}{2} x^2 \qquad u = p - kA$$

$$A_j = -\frac{1}{2} \epsilon_{je} x_e$$

Motivation

$$\dot{J}_1 = (n_2 - n_3) J_2 J_3$$

$$\dot{J}_2 = (n_3 - n_1) J_1 J_3$$

$$\dot{J}_3 = (n_1 - n_2) J_1 J_2$$

Hamiltonian descriptions

$$\{J_1, J_2\} = J_3$$

$$\{J_2, J_3\} = J_1 \qquad H = \frac{1}{2}(n_1 J_1^2 + n_2 J_2^2 + n_3 J_3^2)$$

$$\{J_3, J_1\} = J_2$$

$$\{J_1, J_2\}_1 = n_3 J_3$$

$$\{J_2, J_3\}_1 = n_1 J_1 \qquad H = \frac{1}{2}(J_1^2 + J_2^2 + J_3^2)$$

$$\{J_3, J_1\}_1 = n_2 J_2$$

compatibility $\{J_l, J_m\}_\lambda = \varepsilon_{lmp}(J_p)(1 + \lambda n_p)$

Quantum? try $J_k \to \hat{J}_k$

Dirac's theorem $\{A, BC\} = \{A, B\} C + B \{A, C\}$

$\Rightarrow \{A, B\} = \lambda (A \cdot B - BC)$

Main tool

Weyl systems in the Lagrangian setting

*-products and Poisson Brackets

Plan

Alternative Hamiltonian and Lagrangian descriptions for classical systems: the Feynman's problem

Examples

Weyl systems

*-products

Alternative quantum descriptions

Lagrangian and Hamiltonian descriptions

$$\ddot{x}_j = f_j(x, \dot{x})$$

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{x}_j \partial \dot{x}_k} \ddot{x}_k + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}_j \partial x_k} \dot{x}_k - \frac{\partial \mathcal{L}}{\partial x_j} = 0$$

Inverse problem

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{x}_j \partial \dot{x}_k} f_k(x, \dot{x}) + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}_j \partial x_k} \dot{x}_k - \frac{\partial \mathcal{L}}{\partial x_j} = 0$$

Hamiltonian

$$\dot{x}_j = \{H, \ \}$$

$$\ddot{x}_j = \{H, \{H, x_j\}\} = f_j(x_j, \{H, x_j\})$$

search for solutions $(\{\ ,\ \}, H)$
on a bundle $E \to Q$

Remark
Equations linear in $\mathcal{L}$
not linear in $\{\ ,\ \}, H$

Feynman's problem
Wigner Phys Rev 77, 711 (1950)
Schwebsr Phys Rev 78, 613 (1950)

Feynman

$E = TQ \quad , \{X_j, X_k\} = 0$

With these conditions, the problem is (locally) equivalent to the Lagrangian problem

By making the additional requirement

$\frac{\partial}{\partial v_j}\{\,,\,\} = 0$ the proof is very simple

from $v_k = \{H, x_k\}$

$$\delta_{jk} = \frac{\partial^2 H}{\partial v_k \partial v_j}\{v_j, x_k\}$$

Extension to internal variables

Alternative descriptions for the isotropic Harmonic Oscillator

$\mathbb{R}^{2n} \equiv \mathbb{C}^n \qquad \Gamma = \sum_k p_k \frac{\partial}{\partial q_k} - q_k \frac{\partial}{\partial p_k}$

$\mathbb{C}^n - \{0\} / (\text{dynamics}) \equiv \mathbb{C}P^{n-1} \times \mathbb{R}$

$\pi : \mathbb{C}^n - \{0\} \rightarrow \mathbb{C}P^{n-1} \times \mathbb{R}$

Define $\theta = \sum_k \frac{p_k dq_k - q_k dp_k}{p_1^2 + p_2^2 + \cdots + p_n^2 + q_1^2 + q_2^2 + \cdots + q_n^2}$

$\omega_f = d(f\theta) + \pi^* \omega$

provides a possible description if

$df \wedge \omega^{n-1} \neq 0 \qquad f, \omega$ on $\mathbb{C}P^{n-1} \times \mathbb{R}$

Subset of "Lagrangian" 2-forms

$dq_j \wedge dq_k \wedge \omega_f^{n-1} = 0 \qquad \forall j, k$

The central force problem

$f_k = -c \frac{q_k}{r^3}$

$\mathcal{L}_\lambda = \mathcal{L}_0 + \lambda \frac{J}{r^2} \qquad r \in \mathbb{R}$

$J = | (\vec{v} \wedge \vec{r})^2 |^{1/2}$

central force (continuation)

$$E_\lambda = E_0 = \frac{1}{2}(\vec{v})^2 + U(r)$$

$$\vec{p} = \frac{\partial \mathcal{L}_\lambda}{\partial \vec{v}} = \vec{v} + \frac{\lambda}{J r^2}\left(\vec{v}\, r^2 - (\vec{r}\cdot\vec{v})\vec{r}\right)$$

solving for $\vec{v}$

$$\vec{v} = \vec{p} - \frac{\lambda}{\tilde{J} r^2}\left(\vec{p}\, r^2 - (\vec{p}\cdot\vec{r})\,\vec{r}\right)$$

$$\tilde{J} = |\vec{p}\wedge\vec{r}|$$

and $\tilde{J} = J + \lambda$

$$H_\lambda = H_0 - \lambda\frac{\tilde{J}}{r^2} + \frac{\lambda^2}{2r^2}$$

Comment

$$H_0 = \frac{1}{2m}\left(\frac{\vec{p}\cdot\vec{r}}{r}\right)^2 + \frac{L^2}{2mr^2} + U(r)$$

unfolding $J+\lambda$

K. S. covering

The inverse probeem for linear systems

$$A = \Lambda H$$

Weyl Systems

(V, ω)

$$W : V \to \mathfrak{U}(\mathcal{H})$$

$$W(u)W(u')W^{+}(u)W^{+}(u') = e^{i\omega(u,u')}$$

Remark

ω symplectic structure on V

define $i_X \omega = -dp$ $\quad i_Y \omega = dq$

X, Y generate an action of $\mathbb{R}^{2n}$

$$W(v_1)W(v_2) = e^{i\frac{\omega}{2}(v_1, v_2)} W(v_1 + v_2)$$

von Neumann

$$V = L \oplus L^* = T^*L$$

$$\mathcal{H} = \mathcal{L}^2(L, \mu_L) \qquad (\alpha, x) \in T^*L$$

$$(W(\alpha, x)\psi)(y) = e^{i\alpha(y)}\psi(x+y)$$

Stone's theorem

$$W(v) = e^{iR(v)}$$

$$J : V \to V \qquad J^2 = -\mathbb{1}$$

$$a(v) = \frac{R(v) + i R(Jv)}{\sqrt{2}}$$

$$a^{+}(v) = \frac{R(v) - i R(Jv)}{\sqrt{2}}$$

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Starting with a projective representation

$$W(v_1)W(v_2) = m(v_1, v_2)\, W(v_1+v_2)$$

$$m(v_1, v_2)\, m(v_2, v_1)^{-1} = \mu(v_1, v_2)$$

$$m(v_1, v_2)\, m(v_2, v_1) = s(v_1, v_2)$$

Bargmann

$$\mu(v_1, v_2) = e^{i\,\omega(v_1, v_2)}$$

$s(v_1, v_2)$ trivial one

Weyl ordering $\rightarrow$ symplectic structur

$$\xi(x_1, \alpha_1, a_1; x_2, \alpha_2, a_2) =$$

$$= \sum_a \nu_a \left[\alpha_{1a}(x_{2a}) - \alpha_{2a}(X_{1a})\right]$$

Additional structures from a polarization

$g((\alpha, x), (\alpha_1, x_1)) = \alpha(x_1) + \alpha_1(x)$ (double Lie)

from Darboux

$$J = dq \otimes \frac{\partial}{\partial p} - dp \otimes \frac{\partial}{\partial q}$$

$$g = dp \otimes dp + dq \otimes dq$$

$$\Delta = p \frac{\partial}{\partial p} + q \frac{\partial}{\partial q}$$ (affine versus linear)

Action of $\mathbb{R}^{2n}$ on V $\quad \phi : \mathbb{R}^{2n} \times V \to V$

$i_X \omega = df_X \qquad i_Y \omega = df_Y$

associative product

$$(h * g)(z) = \int (\phi_X^* h)(z)(\phi_Y^* g)(z)\, e^{i\hbar\{f_X, f_Y\}} dx\,dy$$

After this preparation, we consider systems with alternative descriptions

TE the tangent bundle of an affine space

$$\omega_{\mathcal{L}} = d\left(\frac{\partial \mathcal{L}}{\partial v_j}\right) \wedge dq_j$$

$$\Phi_{\mathcal{L}} : \mathbb{R}^{2n} \times TE \to TE$$

$$i_X \omega_{\mathcal{L}} = -d\left(\frac{\partial \mathcal{L}}{\partial v_j}\right) \qquad i_Y \omega_{\mathcal{L}} = dq_j$$

$$E_{\mathcal{L}} = v_j \frac{\partial \mathcal{L}}{\partial v_j} - \mathcal{L}$$

Weyl system associated with $\omega_{\mathcal{L}}$

$$i\hbar \frac{d}{dt} f = E_{\mathcal{L}} *_{\mathcal{L}} f - f *_{\mathcal{L}} E_{\mathcal{L}}$$

Remark:

Polarization and linear structure depending on $\mathcal{L}$

Free action of $\mathbb{R}^{2n}$

Dynamics : linear map versus derivation

Spectrum associated with the linear map

Symplectic transformations and metaplectic representation

$$\begin{tikzcd}
(V,\omega) \arrow[r, "W"] \arrow[d, "S"] & \mathfrak{A}(\mathcal{H}) \arrow[d, "\nu_S"] \\
(V,\omega) \arrow[r] & \mathfrak{A}(\mathcal{H})
\end{tikzcd}$$

$$\nu_S(W(v)) = U_S^{+}\, W(v)\, U_S = W(Sv)$$
$$= e^{i U_S R(v) U_S}$$

Infinitesimal symplectic transformations are derivations for the pointwise product on $\mathcal{F}(V)$ and the $*_\omega$-product

Reduction and anomalies

An example

$$\frac{d}{dt} q = p \qquad \omega = dp \wedge dq$$

$$\frac{d}{dt} p = -q$$

$$P = p\,(1 + f(p^2+q^2)) \qquad Q = q\,(1 + f(p^2+q^2))$$

$$\frac{d}{dt} Q = P \qquad \frac{d}{dt} P = -Q$$

$$\omega' = dP \wedge dQ = G(p^2+q^2)\, dp \wedge dq$$

The dynamics is linear with respect to alternative linear structures on $V \; (\equiv \mathbb{R}^2)$

Lagrangian subspaces and Lagrangian submanifolds

$\Rightarrow$ alternative quantum structures:

the same vector fields preserves different Hermitian products, alternative $*$-products $\Rightarrow$ alternative products on the space of observables dynamics: inner derivation for two different products

Quantum counterpart

$$\frac{da}{dt} = -i a \qquad \frac{d}{dt} a^+ = i a^+$$

Define

$$A = a f(a^+a) \qquad B = f(a^+a)a$$

$$\frac{d}{dt} A = -i A \qquad \frac{d}{dt} A^+ = i A^+$$

$$AA^+ - A^+A = \phi(F^{-1}(\hat{N}))$$

$$\hat{n} = F^{-1}(\hat{N}) \qquad F(\hat{n}) = f^2(\hat{n})\hat{n}$$

$$\phi(x) = (x+1)f^2(x+1) - x f^2(x)$$

From $a|0\rangle = 0$ $\qquad A|0\rangle = 0$

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}}|0\rangle \qquad |N\rangle = \frac{(A^+)^n}{\sqrt{n!}}|0\rangle$$

we have two scalar products

$$(n|m)_1 = \delta_{nm} \qquad (N|M)_2 = \delta_{n,m}$$

$$a = \sum_n |n-1\rangle \sqrt{n} \langle n| \qquad A = \sum_n |n-1\rangle f(n)\sqrt{n} \langle n|$$

The notion of adjoint changes

$$B^+ = a \frac{1}{f(\hat{n})} \qquad [B^+, B] = 1$$

a "non-linear" realization of the Heisenberg algebra

The inverse problem for Quantum dynamics

Starting with

$$i\hbar \frac{d}{dt}\psi = H\psi$$

i) Search for scalar products on states $h(\psi,\phi)$ such that

$$\frac{d}{dt} h(\psi,\phi) = h\left(\frac{d}{dt}\psi, \phi\right) + h\left(\psi, \frac{d\phi}{dt}\right)$$

ii) $i\hbar \frac{d}{dt} A = \Gamma(A)$

search for associative products on observables such that Γ becomes an inner derivation

Examples

- $(\psi|\varphi)_K = (\psi| e^{\lambda K} \varphi)$
- $A \underset{K}{\cdot} B = A e^{\lambda K} B$

$$[H,K] = 0$$

Classical limit

$$A \underset{K}{\cdot} B = A\, e^{\lambda K} B$$

$$f_A(v) = \mathrm{Tr}\, A\, W(v)$$

$$f_A *_K f_B = f_A * f_\lambda * f_B =$$

$$= f_A f_\lambda f_B + i\frac{\hbar}{2} f_\lambda \{f_A, f_B\} + i\frac{\hbar}{2} f_A \{f_\lambda, f_B\} +$$

$$+ i\frac{\hbar}{2}\{f_A, f_\lambda\} f_B + O(\hbar^2).$$

Jacobi bracket instead of a Poisson Bracket

Remark :

All of these deformations give rise to compatible Poisson Brackets

We used a general study of associative products

Consider

$$\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \qquad (A,B) = AB$$

a linear map $N : \mathcal{A} \to \mathcal{A}$

$$\mu_N : (A,B) = A \circ_N B = N(A)B + AN(B) - N(AB)$$

defines a new algebra structure $(\mathcal{A}, \mu_N)$

We introduce a tensor T_N

$$T_N(A,B) = N(A \circ_N B) - N(A)N(B)$$

Thm : μ_N is associative iff

$$A T_N(B,C) - T_N(AB,C) + T_N(A,BC) - T_N(A,B)C = 0$$

moreover

$$\mu + \lambda \mu_N$$

defines a family of associative products

- Alternative commutation relations: physical consequences?

- Reduction and unfolding procedures: symplectic realizations of Poisson manifolds

- Constraints and Zeno dynamics