Noncommutative Geometry & Gravity

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1 General introduction and motivation

History:

Heisenberg *et al.* and the lattice (1930's)

Snyder and 'fuzz' (1947); Lorentz invariance

von Neumann and 'noncommutative geometry' (1950's)

Connes and 'noncommutative differential geometry' (1980's)

Witten and the string connection (2000)

Mathematics:

Formulate as much as possible the geometry of a manifold V in terms of an algebra $\mathcal{C}(V)$ of complex-valued functions (smooth, continuous, measurable, all functions) (Koszul 1960's)

Replace the algebra $\mathcal{C}(V) \mapsto \mathcal{A}$ by a noncommutative algebra $\mathcal{A}$ (associative, with unit and with an involution $*$)

Since V as a manifold of dimension m can be embedded in $\mathbb{R}^n$ for some $n > m$, choose $\mathcal{A}$ defined in terms of n generators and $n - m$ relations

Represent the algebra $\mathcal{A}$ by an algebra of operators on a Hilbert space

Physics:

Practical physics: introduce a cut-off Λ; points 'fuzzy' to order Λ^{-1}

'Fundamental' Physics: replace points by 'Planck cells'; no UV divergences

Solid-state analogy: Coordinates as order parameters; course-graining

Related things: random lattices, qantum nets, twistors, Sakarov's induced gravity, Wheeler's graviton as phonon *et cætera*

Unrelated things: Schrödinger's position operators

$$\mathbf{q}(t) = \mathbf{x}(t) + m^{-2}\mathbf{S} \times \mathbf{p}(t)$$

to explain the Zitterbewegung of an electron

Comparison with quantum mechanics: particle in a plane: (x^1, x^2, p_1, p_2)

a) Classical mechanics: 4 commuting operators

b) Quantum mechanics: $[x^i, p_j] = \hbar\delta^i_j$;
'Bohr cells' of area $2\pi\hbar$

c) Magnetic field B normal to the plane:
$[p_1, p_2] = i\hbar eB$; 'Landau cells' of area $\hbar eB$;
IR cut-off: $p^2 \gtrsim \hbar eB$

d) Gaussian curvature K: $p^2 \gtrsim \hbar^2 K$

e) 'Quantized' coordinates: $x^i \mapsto q^i$;
$[q^1, q^2] = i\kbar q^{12}$; 'Planck cells' of area $2\pi\kbar$;
UV cut-off: $p^2 \lesssim \hbar^2/\kbar$

Situations d) and e) together imply:

$$I = \int \frac{dp_1 dp_2}{p_1^2 + p_2^2} \sim \log(\kbar K)$$

Let ϕ_r be eigenmodes of an operator Δ,

The 2-point function $G \in \mathcal{H} \otimes \mathcal{H}$:

$$G = \sum_r \lambda_r^{-1} \phi_r \otimes \phi_r^*$$

Suppose $\mathcal{A} = \mathcal{A}(x^1, x^2)$ and rewrite

$$x^\mu \otimes 1 = \bar{x}^\mu + \delta x^\mu, \qquad 1 \otimes x^\mu = \bar{x}^\mu - \delta x^\mu$$

Then if J^{12} is in the center of the algebra

$$[\bar{x}^\mu, \delta x^\nu] = 0, \quad [\bar{x}^1, \bar{x}^2] = [\delta x^1, \delta x^2] = \frac{1}{2} i k J^{12}$$

There can be no state $|0\rangle$ (on the diagonal) with

$$\delta x^1 |0\rangle = 0, \quad \delta x^2 |0\rangle = 0$$

Conjecture: J^{12} determines (in Wheeler's language) the 'lattice' spacings away from (flat space) equilibrium and determines what we call the (classical) 'gravitational' field

2 Noncommutative geometry and gravity

The euclidean classical action:

$$S[g] = \Lambda_c \mathrm{Vol}(V)[g] + \mu_P^2 \int_V R + \cdots$$

The euclidean quantum action:

$$\Gamma[g] = S[g] + \frac{1}{2}\hbar \mathrm{Tr}\, \log \Delta[g] \simeq z_0 \mu_P^4 \mathrm{Vol}(V)[g]) +$$

$$z_1 \mu_P^2 \int_V R + z_2 (\log \frac{\mu_P}{\mu}) S_2[g] + 0(\hbar^2) + \cdots$$

$\Delta[g]$: any mode in a gravitational field g

Sakharov's idea: there is no classical action

Wheeler's idea: 'Gravitation is to particle physics as elasticity is to chemical physics: merely a statistical measure of residual energies.'

Replace (Minkowski) coordinates $\tilde{x}^\mu$ by generators x^μ of a noncommutative algebra $\mathcal{A}_k$ with

$$[x^\mu, x^\nu] = i k J^{\mu\nu}, \qquad k \simeq \mu_P^{-2} = G\hbar$$

Structure of the algebra: $[x^\lambda, J^{\mu\nu}]$ *et cætera*

'Heisenberg' uncertainty relations: $\Lambda^2 k \lesssim 1$

Fuzzy space-time: cells of volume $\simeq (2\pi k)^2$

In the limit $\mu_P \to \infty$: $x^\mu \to z\,\tilde{x}^\mu$

Representation: x^μ become unbounded hermitian operators on some Hilbert space

The whole idea is contained in the diagram

$$\begin{array}{ccc} \mathcal{A}_k & \Longleftarrow & \Omega^*(\mathcal{A}_k) \\ \Downarrow & & \Uparrow \\ \text{Cut-off} & & \text{Gravity} \end{array}$$

The classical Cartan formalism

Special case of 'parallelizable' geometries with

$\tilde{\mathcal{A}} = \mathcal{C}^\infty(V)$: $\Omega^*(V)$: DeRham; $\Omega^1(V)$ free

Frame: $\tilde{\theta}^i(\tilde{e}_j) = \delta^i_j$, $\tilde{e}_i = \tilde{e}^r_i \tilde{\partial}_r$

Spin connection: $\tilde{d}\tilde{\theta}^i = -\frac{1}{2}\tilde{C}^i{}_{jk}\tilde{\theta}^j\tilde{\theta}^k$

Commutation relations: $[\tilde{e}_j, \tilde{e}_k] = \tilde{C}^i{}_{jk}\tilde{e}_i$

Gauge condition: $\tilde{e}_i\tilde{C}^i{}_{jk} = 0$

Petrov's classification

In general the manifold has a Weyl tensor R_{ijkl} and a symplectic stucture: K_{ij}.

The tensor K_{ij} has 2 principle nul directions ξ^i and can be of type N or I

The Weyl tensor has 4 principle null directions ζ^i and can be of type N, D, II or I

The NC Cartan formalism

Find an algebra $\mathcal{A}$ with generators λ_i and a morphism

$$\tilde{e}_j \mapsto \lambda_j$$

with perhaps a central extension.

An example

$V = \mathbb{R}^4$ with $\tilde{\theta}^i = d\tilde{x}^i$ and $\tilde{e}_j = \tilde{\partial}_j$

Commutation relations: $[\tilde{e}_j, \tilde{e}_k] = 0$

$$\tilde{e}_j \mapsto \lambda_j$$

Central extension: $[\lambda_j, \lambda_k] = K_{jk}$

Write $\lambda_i = -K_{ij}x^j$

Momentum generators: λ_i
Position generators: x^j
Fourier transform.

Special case of 'parallelizable' geometries with

$\mathcal{A}$: $*$-algebra; $\Omega^*(\mathcal{A})$: calculus; $\Omega^1(\mathcal{A})$ free

Frame: $\theta^i(e_j) = \delta^i_j, \qquad e_i = \mathrm{ad}\,\lambda_i$

Dictionary:

$$\begin{aligned} \lambda_i &\mapsto \frac{1}{i\hbar} p_\alpha \\ \theta^i &\mapsto dx^\alpha \\ \theta^i(e_j) &\mapsto \frac{1}{i\hbar}[p_\beta, x^\alpha] \end{aligned}$$

Position 'space' and p- 'space' are one 'space'

Dimension $= n =$ rank $\Omega^1(\mathcal{A})$

Reality: $(\lambda_i)^* = -\lambda_i, \qquad (\theta^i)^* = \theta^i$

The NC rotation coefficients

Product π: $\quad \pi(\theta^i \otimes \theta^j) = P^{ij}{}_{kl}\theta^k \otimes \theta^l$

In de Rham's case: $\quad P^{ij}{}_{kl} = \frac{1}{2}(\delta^i_k \delta^j_l - \delta^j_k \delta^i_l)$

Dirac: $\quad \theta = -\lambda_i \theta^i, \qquad df = -[\theta, f]$

The exterior derivative

$$d^2 = 0 \qquad \text{implies} \qquad d\theta + \theta^2 = -\frac{1}{2}K_{ij}\theta^i\theta^j$$

The coefficients

$$C^i{}_{jk} = F^i{}_{jk} - 2\lambda_i P^{(il)}{}_{jk}$$

Constraint equation

$$2P^{ij}{}_{kl}\lambda_i\lambda_j - F^i{}_{kl}\lambda_i - K_{kl} = 0$$

Write $\quad P^{ij}{}_{kl} = \frac{1}{2}(\delta^i_k\delta^j_l - \delta^j_k\delta^i_l) + i\hbar Q^{ij}{}_{kl}$

with $\quad Q^{[ij]}{}_{kl} = Q^{ij}{}_{(kl)} = 0$

The coefficients:

$$C^i{}_{jk} = F^i{}_{jk} - 4i\hbar\lambda_l Q^{il}{}_{jk}$$

Constraint equation:

$$[\lambda_k, \lambda_l] = K_{kl} + F^i{}_{kl}\lambda_i - 2i\hbar\lambda_i\lambda_j Q^{ij}{}_{kl}$$

or:

$$[\lambda_k, \lambda_l] = K_{kl} + \frac{1}{2}(F^i{}_{kl} + C^i{}_{kl})\lambda_i$$

Dictionary:

$$\text{Constraint equation} \quad \mapsto \quad [p_\alpha, p_\beta] = 0$$

3 Models

3.1 Kluza-Klein

The example $\mathcal{A} = \mathcal{C}(V) \otimes M_n$:

$$\Omega^1(M_n) \simeq \bigoplus_1^d M_n, \quad n >> d$$

Differential calculus: $\Omega^*(\mathcal{A}) = \Omega^*(V) \otimes \Omega^*(M_n)$

Therefore $\Omega^1(\mathcal{A}) = \Omega^1_h \oplus \Omega^1_v$ with

$$\Omega^1_h = \Omega^1(V) \otimes M_n, \qquad \Omega^1_v = \mathcal{C}(V) \otimes \Omega^1(M_n)$$

We have $F\psi = D^2\psi$ where

$$F = \frac{1}{2} F_{ij}\theta^i\theta^j = \frac{1}{2} F_{\alpha\beta}\theta^\alpha\theta^\beta + D_\alpha\phi_b\theta^\alpha\theta^b + \frac{1}{2} F_{ab}\theta^a\theta^b$$

with $\Omega_{ab} = [\phi_a, \phi_b] - C^c{}_{ab}\phi_c$

The electromagnetic action for (A, ϕ) is

$$S[A, \phi] = \frac{1}{4}\mathrm{Tr} \int F_{\alpha\beta} F^{\alpha\beta}$$

$$+ \frac{1}{2}\mathrm{Tr} \int D_\alpha \phi_a D^\alpha \phi^a - \int V(\phi)$$

with $\quad V(\phi) = -\frac{1}{4}\mathrm{Tr}\,(\Omega_{ab}\Omega^{ab})$

If $d = 3$ the (hidden) 'quantum cell' has area

$$2\pi\hbar \simeq \frac{1}{n} 4\pi r^2$$

The potential $V(\phi)$ vanishes when ϕ lies on a gauge orbit of a representation of SU_2

There are $\quad p(n) \simeq \dfrac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}$ such orbits

The gravitational action is Einstein-Hilbert in 'dimension' $4 + d$ (plus Gauss-Bonnet terms)

3.2 Lobachevsky plane

Let $V = \{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 \mid \tilde{y} > 0\}$

A moving frame is given by

$\theta^1 = \tilde{y}^{-1} d\tilde{x}, \ \ \theta^2 = \tilde{y}^{-1} d\tilde{y}, \quad ds^2 = \tilde{y}^{-2}(d\tilde{x}^2 + d\tilde{y}^2)$

Introduce $\mathcal{A}_h$ with hermitian generators (x, y) and relation

$$[x, y] = -2ihy$$

A frame is given by

$$\theta^1 = y^{-1} dx, \qquad \theta^2 = y^{-1} dy$$

The structure of $\Omega^*(\mathcal{A})$ is given by

$$(\theta^1)^2 = 0, \qquad (\theta^2)^2 = 0, \qquad \theta^1\theta^2 + \theta^2\theta^1 = 0$$

This algebra and differential calculus are invariant under the coaction of the Jordanian deformation of SL_2;

3.3 $\mathbb{R}^3_q$

We set $x^a = (x^-, y, x^+), \quad h = \sqrt{q} - 1/\sqrt{q}$

The defining relations are

$$x^- y = q\, y x^-,$$

$$x^+ y = J^{-1} y x^+,$$

$$[x^+, x^-] = h y^2$$

Consider the elements $\lambda_a \in \mathbb{R}^3_q$ with

$$\lambda_- = +h^{-1} q \Lambda y^{-1} x^+,$$

$$\lambda_0 = -h^{-1} \sqrt{q} \Lambda y^{-1} r,$$

$$\lambda_+ = -h^{-1} \Lambda y^{-1} x^-$$

The 1-forms θ^a dual to the derivations $e_a = \mathrm{ad}\, \lambda_a$ constitutes a frame

Commutation relations are identical to those of x^a

3.4 Kasner metric

A classical moving frame:

$$\theta^0 = dt, \qquad \theta^a = dx^a - p^a_b x^b t^{-1} dt$$

Dual to the derivations

$$e_0 = \partial_t + p^i_j x^j t^{-1} \partial_i \qquad e_a = \partial_a$$

Lie-algebra structure

$$[\tilde{e}_a, \tilde{e}_0] = \tilde{C}^b{}_{a0} \tilde{e}_b, \qquad [\tilde{e}_a, \tilde{e}_b] = 0$$

with

$$\tilde{C}^b{}_{a0} = p^b_a t^{-1}$$

Curvature form given by

$$\tilde{\Omega}^a{}_0 = (p^2 - p)^a_b t^{-2} \tilde{\theta}^0 \tilde{\theta}^b,$$
$$\tilde{\Omega}^a{}_b = -\frac{1}{2} p^a_{[c} p_{d]b} t^{-2} \tilde{\theta}^c \tilde{\theta}^d$$

If $p = \mathrm{diag}(p_1, p_2, p_3)$ a family of Ricci-flat metrics

$$p_1 = \frac{1+\omega}{1+\omega+\omega^2}, \qquad p_2 = \frac{\omega(1+\omega)}{1+\omega+\omega^2},$$
$$p_3 = -\frac{\omega}{1+\omega+\omega^2}.$$

The most interesting value is $\omega = 1$ in which case

$$p_1 = p_2 = \frac{2}{3}, \qquad p_3 = -\frac{1}{3}.$$

Other distinguished values for ω. For example the p_a are given by

$$p_a = (\epsilon - \epsilon^2, -\epsilon, 1 + \epsilon^2) + o(\epsilon^3)$$

when $\epsilon = \omega + 1 \to 0$.

The non-vanishing components of the torsion-free connection form are given by

$$\tilde{\omega}^0{}_a = -\tilde{g}_{ab}\tilde{\omega}^b{}_0, \qquad \tilde{\omega}^a{}_0 = p^a_b t^{-1}\tilde{\theta}^b$$

The conjugacy relations which define the Kasner metric are given by

$$[\lambda_0, t] = 1, \quad [\lambda_0, x^b] = p^b_c f \mu x^c,$$

$$[\lambda_a, t] = 0, \quad [\lambda_a, x^b] = \delta^b_a.$$

The $f = f(t)$: generalization of $(t\mu)^{-1}$

Assume the same functional form, as the commutative limit.

Correspondence principle $\tilde{e}_\alpha \mapsto e_\alpha$

Central extension

$$[\tilde{\lambda}_\alpha, \tilde{\lambda}_\beta] \mapsto [\lambda_\alpha, \lambda_\beta] + K_{\alpha\beta}$$

Introduce

$$l_a = i\bar{k} K_{0a}, \qquad k^a = \frac{1}{2} i\bar{k} \epsilon^{abc} K_{bc}$$

The metric and symplectic structure have a Petrov type

The 'correspondence principle' can be written as

$$C^a{}_{b0} = F^a{}_{b0} - 4i \lim_{\hbar \to 0} \hbar \lambda_c Q_+^{ca}{}_{b0}.$$

Commutative limit: $4i\hbar\lambda_d Q_+^{ad}{}_{0b} \to -p_b^a t^{-1}$

In an adapted frame $k^a = (0, 0, k)$ and $l_a = (0, 0, -l)$. the commutation relations are, with $p_1 + p_2 = -2\gamma k l$.

$$[\lambda_1, \lambda_2] = (i\hbar)^{-1} k(1 - \tfrac{1}{2}\gamma f^2),$$

$$[\lambda_0, \lambda_1] = \tfrac{1}{2} p_1 \mu \lambda_1 f,$$

$$[\lambda_0, \lambda_2] = \tfrac{1}{2} p_2 \mu \lambda_2 f,$$

$$[\lambda_0, \lambda_3] = (i\hbar)^{-1}(l - \tfrac{1}{2} q f^2).$$

The generators λ_0 and λ_3 form a closed subalgebra.

It follows that $e_a f = 0$ which implies that

$$\dot{f} + \frac{1}{2}\mu q f^2 + c = 0, \qquad c = -\mu l_a k^a = \mu l k.$$

This equation is invariant under the transformation

$$f \mapsto -f^{-1}, \qquad \frac{1}{2}\mu q \mapsto c, \qquad c \mapsto \frac{1}{2}\mu q.$$

Suppose that $c > 0$. Then a solution to is given by

$$f(t) = \sqrt{\frac{2kl}{q}} \cot\left(\sqrt{\frac{klq}{2}}\mu t\right)$$

Near the point where it vanishes

$$(f(t))^{-1} \simeq \sqrt{\frac{2kl}{q}}\sqrt{\frac{klq}{2}}\mu t = kl\mu t$$

If we compare with classical limit we conclude that $kl = 1$. The solution can be written then as

$$f(t) = \sqrt{\frac{2}{q}} \cot\left(\sqrt{\frac{q}{2}}\mu t\right)$$

Near the origin it behaves as

$$f(t) \simeq 2(\mu t)^{-1}.$$

If one ajusts parameters so that in the region of space-time where the field is weak the solution coincides with the classical one then near the singularity it is renormalized by a factor $2/q$.

We must impose $p^a_a = 3q$: $\quad q = \frac{1}{3}, \quad \gamma = -\frac{1}{3}$

To find the x^μ solve the conjugacy relations
Introduce the 'canonical transformation' $\lambda_\alpha \mapsto \mu_\alpha$ defined by

$$
\begin{aligned}
&\lambda_1 = \mu_1, \qquad \lambda_2 = \mu_2, \\
&i\bar{k}\mu\lambda_3 = \sqrt{\frac{q}{2}} \cot\left(\sqrt{\frac{q}{2}} i\bar{k}\mu\mu_3\right), \\
&\lambda_0 = \mu_0.
\end{aligned}
$$

Fourier transform given by

$$
\begin{aligned}
&x = -i\bar{k}\mu_2, \qquad y = i\bar{k}\mu_1 \\
&z = -\frac{1}{2} i\bar{k}\Big(\lambda_0 \arcsin^2(a\mu_3) + \arcsin^2(a\mu_3)\lambda_0\Big) \\
&t = i\bar{k}\mu_3
\end{aligned}
$$