LOCAL CONFORMAL NETS

conformal QFT has several motivations, e.g.:

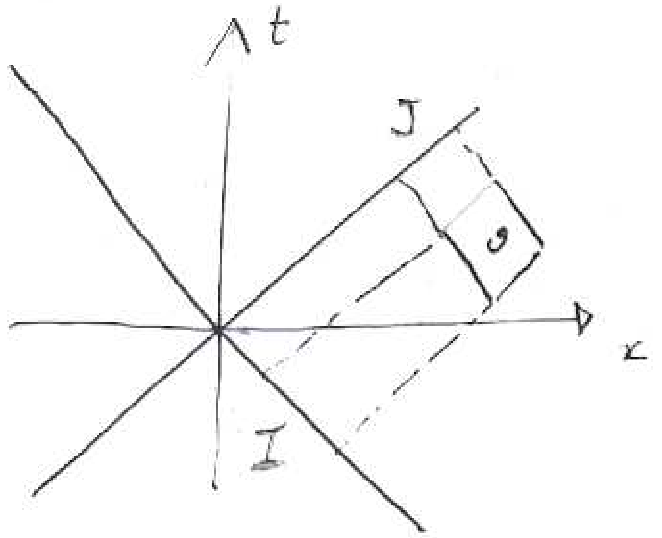
- low dim critical phenomena
- topological QFT
- holography
- string theory
- algebraic QFT
- quantum black holes

and various descriptions, e.g.

- vertex operators
- Kac-Moody algebras
- Wightman fields
- W-algebras

What is the common feature?
when do two models contain the same information
Can we classify CQFT?

Chiral conformal QFT



$\Phi(x,t)$ 2-dim QFT

$$(x,t) \mapsto \left(\frac{x}{x^2-t^2}, \frac{t}{x^2-t^2}\right)$$

symmetry

Example: free scalar field

$$\Box\phi = 0 \Rightarrow \Phi = \Phi_1(x-t) + \Phi_2(x+t)$$

chiral components

$$A(\mathcal{O}) = A_1(I) \otimes A_2(J)$$

where $A_i(I) = \{ e^{i\phi(f)}, \operatorname{supp} f \subset I \}''$

In general

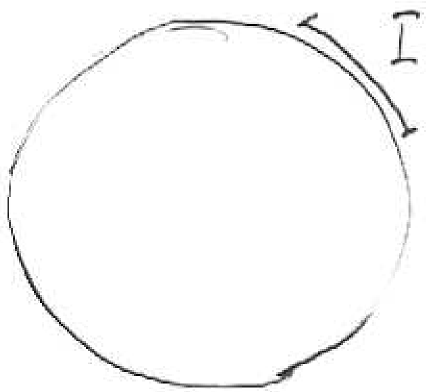
$$A(\mathcal{O}) \supset A_1(I) \otimes A_2(J)$$

(finite-index) inclusion

Conformal invariance $\Rightarrow$ A_i net on S^1

CONFORMAL NETS

A (local) Möbius covariant net (or conformal precosheaf) of v.N. algebras is a map



$$I \to \mathcal{A}(I) \subset \mathcal{B}(\mathcal{H})$$

interval — fixed Hilbert space

s.t. Isotony: $I_1 \subset I_2 \Rightarrow \mathcal{A}(I_1) \subset \mathcal{A}(I_2)$

Möbius invariance: $\exists\, U$ unitary rep. of $PSL(2,\mathbb{R})$ on $\mathcal{H}$ s.t.

$$U(g)\,\mathcal{A}(I)\,U(g)^{-1} = \mathcal{A}(gI)$$

Positive energy: $L_0 \geq 0$

$$L_0 \equiv i\frac{d}{d\theta}U(R(\theta))\Big|_{\theta=0} \quad \text{(conf. Hamiltonian)}$$

Existence of the vacuum $\exists\, \Omega \in \mathcal{H}$

$\mathcal{U}(g)\Omega = \Omega$, $\quad \Omega$ cyclic for $\cup A(I)$

Irreducibility $\left(\cup A(I)\right)'' = B(\mathcal{H})$

$\Longleftrightarrow$ Ω unique $\mathcal{U}$-invariant vector

Locality: $[A(I_1), A(I_2)] = \{0\}$, $I_1 \subset I_2'$

CONSEQUENCES:

Reeh-Schlieder thm. Fredenhagen-Joeß

$$\overline{A(I)\Omega} = \overline{A(I)'\Omega} = \mathcal{H}$$

Bisognano-Wichmann property

$$\begin{cases} \Delta_I^{it} = U(\Lambda_I(2\pi t)) \\ J_I = \text{reflection } I \to I' \end{cases}$$ geometric actions

Hislop, L. '82 (Wightman theory)

Brunetti, Guido, L. '93

Fröhlich, Gabbiani '93

independent

Haag duality $\quad A(I') = A(I)'$

Factoriality $\quad A(I)$ is a factor of type III_1 in Connes' classification

Additivity $\quad \{I_i\}, I$ intervals

$I \subset \bigcup I_i \quad = \quad A(I) \subset \bigvee_i A(I_i)$

Fredenhagen - Jorß

DOPLICHER-HAAG-ROBERTS REPRESENTATIONS OF A

(superselection sectors) DHR $\to$ conformal

A representation of A is a map

I interval of $S^1 \longrightarrow \pi_I \in Rep(A(I))$,

of rep's on a fixed separable Hilbert space s.t.

$$\pi_{\tilde{I}}\big|_{A(I)} = \pi_I \quad \text{if } I \subset \tilde{I}$$

π is covariant is there exists a unitary rep of $SL(2,\mathbb{R})^\sim$, U_π, :

$$U_\pi(g)\,\pi(X)\,U_\pi(g)^{-1} = \pi(U(g)XU(g)^{-1})$$

$X \in \cup A(I)$.

By Takesaki-Feldman (recall: $A(I)$ is type III_1 factor) every rep. of A is <u>locally normal</u>:

π unitarily equiv. to id_I

As a consequence

π rep of A $\simeq$ ρ DHR endom.

ρ localized in I :

$\rho \restriction A(I_1) = id$ if $I_1 \subset I'$

$\rho(A(\tilde{I})) \subset A(\tilde{I})$ if $\tilde{I} \supset I$

There is a universal C^*-algebra $C^*(A)$ s.t.

$\pi \in Rep(C^*(A)) \simeq \pi = \pi_0 \circ \rho$

$\rho \in End(A)$ Fredenhagen

Rep(A) is a <u>tensor</u> C^*-<u>category</u> : $\rho \otimes \sigma \equiv \rho\sigma$ with <u>unitary</u> <u>braiding</u>

$$\varepsilon(\rho,\sigma) \in (\rho\sigma, \sigma\rho)$$

$$\rho \quad \sigma \xrightarrow{\varepsilon(\rho,\sigma)} \sigma \quad \rho$$

As a consequence

π rep of A $\approx$ ρ DHR endom.

ρ localized in I :

$\rho \restriction A(I_1) = id$ if $I_1 \subset I'$

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INDEX-STATISTICS THM L. '88

CONFORMAL SPIN-STATISTICS Guido,L. '96

$$\lambda_\rho = \frac{\omega_\rho}{\sqrt{\mathrm{Ind}(\rho)}}$$

$$\boxed{\begin{array}{c}\text{DHR}\\ \text{statistics}\\ \text{parameter}\end{array}} = \boxed{\frac{\text{univalence}}{\sqrt{\text{Jones index}}}}$$

- λ_ρ classifies statistics in dim 4
  DHR '74

- $\omega_\rho = e^{i2\pi L_0}$ "spin"

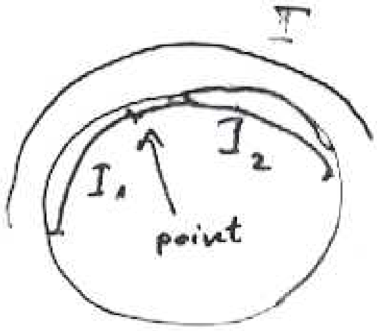
- Jones index is minimal
  index of $\rho(A(I)) \subset A(I)$

STRONG ADDITIVITY

A local conformal net on S^1 is strongly additive if

$$A(I_1) \vee A(I_2) = A(I)$$

$I = I_1 \cup I_2 \cup \{\text{point}\}$



Cut S^1 at one point $\{\infty\}$

$$S^1 = \mathbb{R} \cup \{\infty\}$$

PROP. A strongly additive

$$\Downarrow$$

A has Haag duality on $\mathbb{R}$

$$A(I) = A(\mathbb{R} \setminus I)'$$

THM. Guido, Wiesbrock, R.L.
The dual net $A^d(I) \equiv A(\mathbb{R} \setminus I)'$
is always <u>conformal</u> and <u>strongly additive</u>

SPLIT PROPERTY

A net $\mathcal{A}$ is split

if whenever $I_1, I_2 \subset S^1$ are separated

the natural isomorphism of v.N. algebras

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \mapsto \mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$$

$$a_1 a_2 \longmapsto a_1 \otimes a_2$$

Buchholz '74; Doplicher L. '84 ....

D'Antoni, L.

THM.

$$\mathrm{Tr}(e^{-\beta L_0}) < \infty \quad \forall \beta > 0$$

$$\Downarrow$$

$\mathcal{A}$ is split

L_0 = conformal Hamiltonian

Buchholz Wichmann '85; ... D'Antoni, Radulescu, L

Buchholz, D'Antoni, L.

CONFORMAL COVARIANCE

A Möbius covariant local net $\mathcal{A}$ on S^1 is conformal (change of terminology) or diffeomorphism covariant if $\exists$ unit. rep U of $\mathrm{Diff}(S^1)$ (projective)

$$U(g)\mathcal{A}(I)U(g)^{-1} = \mathcal{A}(gI),\ g \in \mathrm{Diff}$$

$$U(g)XU(g)^{-1} = X,\quad g \in \mathrm{Diff}(S^1),\ g(t)=t,\ t \notin I$$

VIRASORO NETS

U ~~irred~~, proj. rep. of $\mathrm{Diff}(S^1)$

$$\mathrm{Vir}(I) \equiv \{U(g),\ g(t)=t,\ t \notin I\}''$$

[Vir nets are labeled by the values of the central charge]

VIRASORO ALGEBRA

$\mathrm{Diff}(S^1) =$ smooth diffeomorphisms of S^1

De Witt = Lie algebra of $\mathrm{Diff}(S^1)$

$$[L_n, L_m] = (n-m)L_{n+m}, \quad L_n = ie^{int}\frac{d}{dt}$$

Cohomology of De Witt is 1-dim:
The Virasoro algebra is the (unique) central extension of De Witt

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n,-m}$$

$$[L_n, c] = 0$$

u Unitary projective rep of $\mathrm{Diff}(S^1)$

$$\Updownarrow$$

Operators L_n, c with relations

$$L^* = L$$

in the representation U

L_0 = conformal Hamiltonian

$L_{-1}, L_0, L_1 = sl(2, \mathbb{R})$

c = central charge

We always assume U has positive energy, i.e. $\boxed{L_0 \geq 0}$

Note: De Witt has no representation with positive energy, but $U = I$

If U is irreducible

$c \in \mathbb{R}$

$h \geq 0$ (spin) lowest eigenvalue of L_0

For any possible value of c and h there is <u>exactly one</u> (unitary, pos. energy) rep. of Vir (e.g. Kac's book)

THM. Friedan, Qiu, Shenker '86

$$C = 1 - \frac{6}{n(n+1)} \quad \text{or} \quad c \geq 1$$

THM. Goddard, Kent, Olive '86

The above values occur

If $c = 1 - \frac{6}{n(n+1)}$ there are

$\frac{n(n-1)}{2}$ rep's. Their spin are

$$h_{p,q} = \frac{((n+1)p - nq)^2 - 1}{4n(n+1)}$$

$1 \leq p \leq n-1, \quad 1 \leq q \leq n$

$(p,q) \sim (n-p,\ n+1-q)$

Representations with the
Same central charge have
fusion (internal product)

Subfactors and Jones index

$N \subset M$ inclusion of factors

$[M:N] < \infty$ i.e. finite Jones index

THM Jones '83

$[M:N] = 4\cos^2 \frac{\pi}{n}$, $n = 3, 4 \ldots$ or $[M:N] \geq 4$

long standing problem
Is there a relation between central charge and Jones index?

THM Popa, Ocneanu '87
$[M,N] < 4$, hyperfinite has

an $A - D_{even} - E_{6,8}$

classification (A-D-case unique
E-case 2 subfactors)

MINIMAL MODELS

$\mathcal{A}$ conformal (= diff. covariant) local net on S^1, U the associative rep of $\mathrm{Diff}(S^1)$

$$\mathcal{A}_{Vir}(I) = \{U(g), g \in \mathrm{Diff}(S^1), g(t) = t, t \notin I\}$$

then $\mathcal{A}_{Vir} \subset \mathcal{A}$ is a subnet

and $\mathcal{A}_{Vir}$ is isomorphic

to Vir_c for a unique c

central charge of $\mathcal{A}$

$$\mathcal{A}_{Vir}(I)' \cap \mathcal{A}(I) = \mathbb{C}$$

If $c<1$ the possible models are the discrete series

PROP. $\mathcal{A}$ is the discrete series (i.e. $c<1$) then

$$[\mathcal{A}(I) : \mathcal{A}_{Vir}(I)] < \infty$$

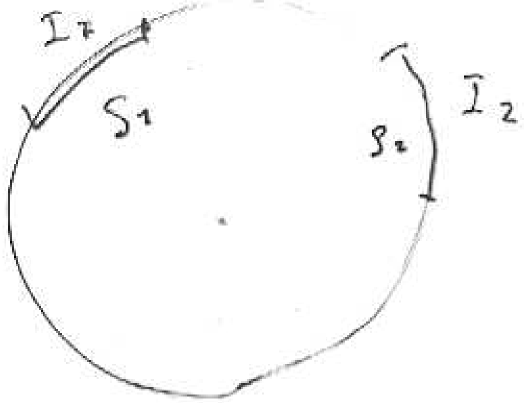
therefore

Classify the discrete series

$$\Updownarrow$$

Classify extension of the Virasoro nets (or Virasoro algebra) with finite Jones index.

FAILURE OF HAAG DUALITY FOR DISCONNETED REGIONS

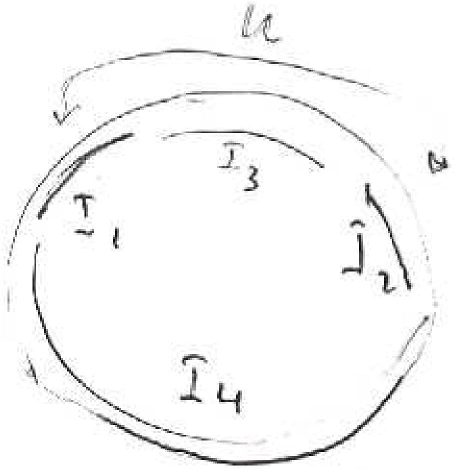


ρ_1 localized in I_1

ρ_2 localized in I_2

$$\rho_1 \simeq \rho_2$$

$$\rho_1 = u\, \rho_2(\cdot) u^*$$



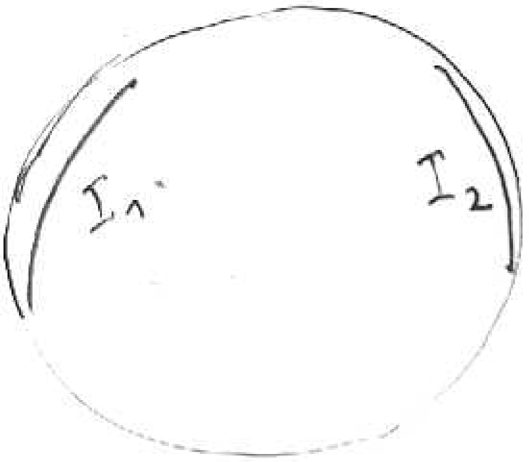
u bilocalized in $I_1 \cup I_2$

$$u \in \left(A(I_3) \vee A(I_4)\right)'$$

$$u \notin A(I_1) \vee A(I_2)$$

The inclusion $\boxed{A(E) \subset A(E')' \equiv \hat{A}(E)}$ contains info on superselection stru.

Is $A(E')'$ generated by $A(E)$ and charge transfers? Is it canonically generated?



$E = I_1 \cup I_2$

$I_2 = j I_1$

$j : z \to -\bar{z}$

ρ localized in I_1

$\bar{\rho} = j \cdot \rho \cdot j$ localized in I_2

$\exists \quad R \in (i, \bar{\rho}\rho)$

$R x = \bar{\rho}\rho(x) R$

$R \in A(E')' = \hat{A}(E)$

$\Rightarrow \quad \bar{\rho}\rho \prec \lambda$, λ dual can. endo of $\hat{A}(E) \to A(E)$

Do all sectors $\bar{\rho}\rho$ sum up to λ?

COMPLETE RATIONALITY

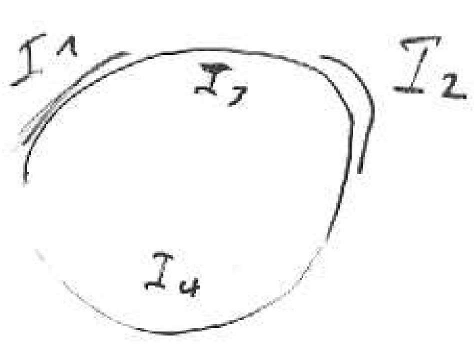
Kawahigashi, Müger, Longo

Let A be a local conformal net on S^1. A is completely rational if

1. A satisfies the split property
2. A is strongly additive
3. $[\hat{A}(E) : A(E)] < \infty$, $E = I_1 \cup I_2$

$$\hat{A}(E) \equiv A(E')'$$

$$\hat{A}(E) \equiv (A(I_3) \vee A(I_4))'$$



Note: 2. $\Leftrightarrow$ Haag duality on $\mathbb{R}$

$$\mu_A \equiv [\hat{A}(E) : A(E)]$$

Lemma: μ_A is independent of E

KAWAHIGASHI, MÜGER, R.L., 2000

THM. $\mathcal{A}$ completely rational conformal net, $E = I_1 \cup I_2 \subset S^1$ a 2-interval.

Then there is a natural isomorphism

$$\boxed{\mathcal{A}(E) \subset \hat{\mathcal{A}}(E) \longleftrightarrow \text{Quantum Double } N \otimes N^{op} \subset M \text{ of all sectors}}$$

$$\mathcal{A}(I_1) \simeq N, \quad \mathcal{A}(I_2) \simeq N^{op}, \quad M \simeq \hat{\mathcal{A}}(E)$$

$$\rho \otimes \rho^{op} \simeq \rho \otimes \bar{\rho} \qquad \bar{\rho} = j \cdot \rho \cdot j$$

In particular: we extend Xu formula in the $SU(N)_k$ models

$$\mu_{\mathcal{A}} = [\hat{\mathcal{A}}(E) : \mathcal{A}(E)] = \sum_i d(\rho_i)^2$$

sum of all sectors, thus $\mathcal{A}$ is rational

indeed

$$\lambda = \bigoplus_i \rho_i \bar{\rho}_i$$

λ dual can. endomorphism $\hat{\mathcal{A}}(E) \to \mathcal{A}(E)$

Thus the sector structure of $\mathcal{A}$ is

Coset models

G compact Lie group

LG loop group

$LG \equiv \{ g : S^1 \to G \}$ pointwise multipl.

$U : LG \to B(\mathcal{H})$ unitary positive energy rep.

i.e. the action $\mathrm{Diff}(S^1) \to \mathrm{Aut}(LG)$ is implemented on $\mathcal{H}$ by a projective pos. energy rep. of $\mathrm{Diff}(S^1)$

U is a vacuum representation if the conformal Hamiltonian (the generator of rotation unitary grp) has a zero eigenvector

Vacuum representation are labeled by a parameter

Let U be the level k vacuum representation of G

$$A(I) = \{g : S^1 \to G \text{ s.t. } \operatorname{supp} g \subset I\}''$$

$I \mapsto A(I)$ conformal net

$A \supset Vir$ or, equivalently, there is a stress energy tensor T_A

$H \subset G$ compact subgroup

$$B(I) = \{g : S^1 \to H \text{ s.t. } \operatorname{supp} g \subset I\}''$$

$I \mapsto B(I) \subset A(I)$ conformal subnet

$T_{G/H} \equiv T_A - T_B$ generate a rep. of Vir (Goddard, Kent, Olive)

$B^c(I) \equiv B' \cap A(I)$ has $T_{G/H}$ stress energy

B^c [illegible]

Complete Rationality is Hereditary

Thm R.L. '01

A (local, irreducible) conformal net

$\mathcal{B} \subset A$ conformal subnet

$[A : B] < \infty$

A completely rational $\Leftrightarrow$ B completely rational

hard part: A compl. rational $\Rightarrow$ B strongly additive
(case $B = A^G$ proved by F. Xu).

Def. [Xu] $B \subset A$ conformal nets

B is cofinite in A if

$$[A : B \vee B^c] < \infty$$

where $B^c(I) = B' \cap A(I)$

Corollary $B \subset A$ conformal nets, B cofinite

A compl. rational $\Leftrightarrow$ both B and B^c are compl. rational

Thm (F. Xu)
The following subnets are cofinite:

(i) $G_{k_1+k_2+\cdots+k_n} \subset G_{k_1} \times G_{k_2} \times \cdots \times G_{k_n}$
diagonal inclusion $k_i \in \mathbb{N}$
$i = 1, \ldots m$, $G = SU(n)$

(ii) $H_{\ell k} \subset G_\ell$ if $H_k \subset G_1$ is
a conformal inclusion
k Dynkin index, $\ell \in \mathbb{N}$, H simple
of type A and G simple.

(iii) $H \subset G_m$ where H is the
Cartan subgroup of G

Corollary The coset models
associated with (i), (ii), (iii)
are completely rational

COR. The braiding associated with <u>all</u> sectors is non-degenerate, namely the tensor category of all representations is modular. →

COR. The Verlinde matrices T and S constructed by Rehren are invertible and give rise to a representation of the modular group $SL(2,\mathbb{Z})$ (relevant in TQFT).

COR. If $E = I_1 \cup I_2 \cup \ldots \cup I_n \subset S^1$ is an n-interval, then $A(E) \subset \hat{A}(E)$ is the "iterated" LR inclusion

$$[\hat{A}(E) : A(E)] = \mu_A^{n-1}$$

$$\lambda = \bigoplus N^{0}_{i_1 \ldots i_n} \, \rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_n$$

COR. The inclusion $A(E) \subset \hat{A}(E)$ depends only on the tensor category of rep's of A, not on its model realization, not on E.

COR. Every sector is sum of sectors with finite dimension. In particular every representation of A is of type I

COR. Every rep. is Möbius covariant with positive energy (by Guido-L.)

COR. Let $A \otimes A^{op} \subset B$ be $(t=0)$ LR net. Then $\mu_B = 0$, thus B has no non-trivial sector (B is a field algebra for $A \otimes A^{op}$).

VIRASORO NETS AS COSET NETS

$\mathrm{Vir}_c =$

coset of

$$SU(2)_{m-1} \subset SU(2)_{m-2} \times SU(2)_1$$

$$c = 1 - \frac{6}{m(m+1)}$$

GKO, Xu, Carpi, Kawahigashi, L.

$\Longrightarrow$

Vir_c is completely rational $c<1$

$\Rightarrow$ Irred. extension of Vir_c have finite index, $c<1$

$\Rightarrow$ structure of sectors. Loke

Q-SYSTEMS R.L. '96

M factor

A Q-systems is a triple

$$(\lambda, V, W)$$

$\lambda \in End(M)$

$V \in Hom(\iota, \lambda)$ (i.e. $Vx = \lambda(x)V$)

$W \in Hom(\lambda, \lambda^2)$ (i.e. $W\lambda(x) = \lambda^2(x)W$)

$$V^* W = \lambda(V)^* W \in \mathbb{R}^+$$

$$\lambda(W)W = W^2$$

THM

Q-system $\Longleftrightarrow$ $\tilde{M} \supset M$, $[\tilde{M} : M] < \infty$

Our problem is to determine all possible Q-systems in Vir_c

MODULAR INVARIANTS

Given a unitary finite-dim. representation U of $SL(2,\mathbb{Z})$ a modular invariant is a matrix $T \in Mat(\mathbb{Z}_+)$ $T_{00}=1$

$$TU = UT$$

VERLINDE-REHREN '90

Rational - with non degenerate braiding $\Rightarrow$ rep of $SL(2,\mathbb{Z})$

COR (of KLM) compl. rational $\rightarrow$ rep of $SL(2,\mathbb{Z})$

CAPELLI, ITZYKSON, ZUBER '87

ADE classification for modular invariants for Vir_c $c<1$

BÖCKENHAUER, EVANS, KAWAHIGASHI
2000

$A \subset B$ conformal nets

$[B : A] < \infty$

α-induction $\Rightarrow$ modular invariant

$$Z_{\mu\nu} = \dim \mathrm{Hom}(\alpha^+_\mu, \alpha^-_\nu)$$

where $\alpha^\pm_\mu$ is the extension from A to B ~~the~~ of a DHR sector μ of A to a $\pm$ solitonic sector of B

(Roberts, Rehren, L. ...)

KAWAHIGASHI, L. classification

THM Let $c = 1 - \frac{6}{m(m+1)}$, $m=2,3\ldots$

The local extensions of Vir_c are classified by A. D_{even}; $E_{6,8}$ pair of graphs with difference of Coxeter number 1

m		
	n	(A_{n-1}, A_n)
$m=$	$4n+1$	(A_{4n}, D_{2n+2})
$m=$	$4n+2$	(D_{2n+2}, A_{4n+2})
	11	$(A_{10}, E_6$
	12	(E_6, A_{12})
	13	(A_{28}, E_8)
	14	(E_8, A_{30})

COROLLARY

Conformal local nets with central charge $c = 1 - \frac{6}{n(n+1)}$ are classified completely by the pair (n, s), s = number of nontrivial subnets

$s = 1$	all n
$s = 2$	$n = 1, 2 \bmod 4$
	$n = 11, 12$
$s = 3$	$n = 29, 30$

PSEUDONETS

A (conformal) pseudonet is a map

$$I \subset S^1 \longmapsto A(I) \subset B(\mathcal{H})$$

from intervals to v. N. algebras s.t.

- Möbius covariance: $\exists\, U : PSL(2,\mathbb{R}) \to B(\mathcal{H})$

  $$U(g)A(I)U(g)^{-1} = A(gI)$$

- Vacuum with Reeh-Schlieder property

  $\exists\, \Omega \in \mathcal{H}$ cyclic and sep. for every $A(I)$ and $U(g)\Omega = \Omega$

- Interval KMS property

  $$\Delta_I^{it} = U(\Lambda_I(-2\pi t))$$

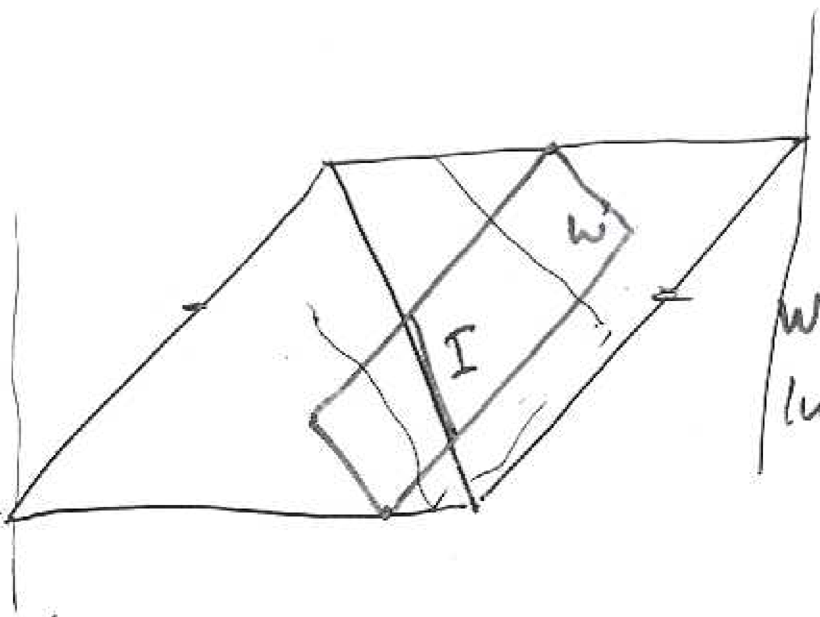
- Locality $[A(I), A(I')] = \{0\}$

dS^2 holography

$\exists$ 1:1 correspondence

local nets on dS^2 $\longleftrightarrow$ local conformal pseudonets on S^1

cyclicity for double cones $\longleftrightarrow$ intersection cyclicity
$$\overline{\left[\bigcap_{I_2 \supset I \supset I_1} B(I)\right]\Omega} = \mathcal{H}$$



Wedge W
Interval I

positive energy $\longleftrightarrow$ pseudonet is a net