INSTANTONS

ON

NONCOMMUTATIVE SPHERES

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Noncommutative manifolds: the instanton algebra and isospectral deformations

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Instantons on the quantum four sphere S^4_q

[illegible]

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Instanton algebras and quantum 4-spheres

[illegible]

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Noncommutative finite dimensional manifolds I. Spherical manifolds and related examples

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STARTING POINT of NCG:

the identification of $\boxed{\text{points}}$ in a space is bound up with the $\boxed{\text{commutativity}}$ of the algebra of coordinates on the space

[illegible] PIECES [illegible]

- Gel'fand - Naimark theorem.

  X a locally compact Hausdorff space
  it can be reconstructed from
  $C_0(X)$ continuous fncts vanishing at ∞
  as the space of characters $\mu : C_0(X) \to \mathbb{C}$

  $\mu(fg) = \mu(f)\,\mu(g)$

  $\mu = \varepsilon_x : f \longmapsto f(x)$ evaluation at some point $x \in X$

- Heisenberg uncertainty principle

  $\Delta q \, \Delta p \geq 4\pi \hbar$ when $[q, p] = i\hbar$

  thus localization at $(q,p) \in \{\text{phase space}\}$ is ruled out

$C_0(X)$ is a commutative C^*-algebra

- C^*-algebras, commutative or not, are the substitutes for topological spaces

To bring in differential geometry one needs to identify a NC analogue of smooth functions

$\mathcal{A} = C^\infty(X)$ is a pre-C^*-algebra

- it is dense in $A = C(X)$
- it is 'stable under holomorphic functional calculus': it is complete in some stronger topology and if $a \in \mathcal{A}$ is invertible in A then $a^{-1} \in \mathcal{A}$

- pre-C^*-algebras, commutative or not, are the substitutes for smooth spaces

for pre C^* alg [illegible]

K_* [illegible] $= K_*$ [illegible]

act as a twisted convolution

$f \in C^\infty(\mathbb{T}^2)$

$$f = \sum_{r_1, r_2} f_{r_1, r_2} \, e^{2\pi i (r_1 x + r_2 y)}$$

pulling back the product from A_θ^2

$$f * g = \sum_{r_1, r_2} (f * g)_{r_1, r_2} \, e^{2\pi i (r_1 x + r_2 y)}$$

$$(f * g)_{r_1, r_2} = \sum_{s_1, s_2} f_{s_1, s_2} \, g_{r_1 - s_1, r_2 - s_2} \, e^{\pi i (r_1 s_2 - r_2 s_1)\theta}$$

a $U(1)$-valued 2-cocycle on $\mathbb{Z}^2$

$$\lambda(r, s) = \exp \pi i (r_1 s_2 - r_2 s_1)\theta$$

the deformed product depends only on its cohomology class in

$H^2(\mathbb{Z}^2, U(1))$

Bundle theory comes from

- Serre - Swann theorem

A vector bundle $E \to X$ is characterized by its sections $\Gamma(E, X)$ which form a $\boxed{\text{module}}$ over $C(X)$

$$\left.\begin{array}{l} s \in \Gamma(E, X) \\ f \in C(X) \end{array}\right. \longmapsto sf \in \Gamma(E, X)$$

a special Kind of module:
projective module of finite type

$$\Gamma(E, X) \oplus (\cdots) = C(X)^N$$

$$\Gamma(E, X) = e\, C(X)^N$$

$$e \in \mathrm{Mat}_N(C(X))$$

$$e^2 = e \quad ; \quad e^* = e$$

- projective modules of finite type over an algebra $\mathcal{A}$ are the substitutes for vector bundles "over $\mathcal{A}$"

Noncommutative spaces

$\updownarrow$

Noncommutative $*$ algebras of functions $\mathcal{A}$

Vector bundles

$\updownarrow$

Projective modules of finite type over $\mathcal{A}$

s.a. idempotent $\quad e \in \mathrm{Mat}_N(\mathcal{A})$

$$e^2 = e = e^*$$

$\mathcal{E} = e\mathcal{A}^N \equiv$ sections of vector bundles

Spaces are constructed out of bundles defined on them

spheres are constructed out of instanton bundles

* $\boxed{\text{instanto}}$ *

a complex vector bundle over a 4-sphere

a [illegible]

higher dimensional generalizations

An Instanton must have the right characteristic classes

$$ch_k(e) = 0 \qquad k = 0, 1$$

$$ch_2(e) \sim \text{volume form}$$

These are components of a Chern-character

$$ch_j(e) = \langle (e - \tfrac{1}{2})(\delta e)^{2j} \rangle \in A \otimes \underbrace{\bar{A} \otimes \ldots \bar{\otimes} \bar{A}}_{2j \text{ times}}$$

$$\bar{A} = A / \mathbb{C}\mathbb{I}$$

They form a cycle in the (b, B) bicomplex of cyclic homology

$$b\, ch_{j+1}(e) = B\, ch_j(e)$$

$$ch_j(e) = \sum (e_{i_0 i_1} - \tfrac{1}{2} \delta_{i_0 i_1}) \otimes e_{i_1 i_2} \otimes \cdots \otimes e_{i_{2j} i_0}$$

- The Hochschild boundary b

$$b(a_0 \otimes a_1 \otimes \dots \otimes a_n) =$$

$$= \sum_{j=0}^{n-1} (-1)^j a_0 \otimes a_1 \otimes \dots \otimes (a_j a_{j+1}) \otimes \dots \otimes a_n$$
$$+ (-1)^n (a_n a_0) \otimes a_1 \dots \otimes a_{n-1}$$

$b^2 = 0$ $\qquad HH_*(A)$ is the homology of this complex

Hochschild homology

- The Connes' boundary B

$$B = B_0 A$$

$$B_0(a_0 \otimes \dots a_n) = \mathbb{1} \otimes a_0 \otimes \dots a_n$$

$$A(a_0 \otimes \dots \otimes a_n) = \frac{1}{n+1} \sum_{j=0}^{n} (-1)^{nj} a_j \otimes a_{j+1} \otimes \dots \otimes a_n \otimes a_0 \otimes \dots \otimes a_{j-1}$$

$$\boxed{bB + Bb = 0}$$

Hochschild homology class for $\mathcal{A} = C^\infty(M)$ correspond to differential forms on M

$$HH_k(C^\infty(M)) \simeq \Omega^k(M)$$

In particular n-cycles c $\quad n = \dim M$

$$bc = 0$$

gives volume forms

There is no general formulae for hochschild cycles of suitable dimension

For spheres we can use the formula for the Chern-character

EVEN S^{2n}

$$ch(e) = \sum_{k=0}^{n} \frac{1}{k!} \, tr\left[e(de)^{2k}\right] = 1 + \frac{i^n (2n)!}{2^{n+1} n!} \mu$$

volume form

e a projection in $Mat_{2^n}(C^\infty(S^{2n}))$

$e^2 = e \quad , \quad e^* = e$

Redefine

$$ch_k(e) \sim tr\left[(e-\tfrac{1}{2})(de)^{2k}\right] \in \Omega^{2k}_{dR}(S^{2n})$$

ODD S^{2n+1}

a similar formula

$$ch_{k+1/2}(u) \sim tr\left[(u^{-1}du)^{2k+1}\right] \in \Omega^{2k+1}_{dR}(S^{2n+1})$$

u a unitary in $Mat_{2^n}(C^\infty(S^{2n+1}))$

$u.u^* = u^*u = 1$

For spheres the top components

$$ch_n(e) \qquad \text{or} \qquad ch_{n+1/2}(u)$$

are Hochschild cycles

$$b\,ch_n(e) = 0 \qquad b\,ch_{n+1/2}(u) = 0$$

In general we shall impose

$$ch_k(e) = 0 \qquad k = 0, 1, \cdots, n-1$$

for e a suitable projection in $\mathrm{Mat}_N(\mathcal{A})$ $\quad \mathcal{A}$ a deformed algebra

or

$$ch_{k+1/2}(u) = 0 \qquad k = 0, 1, \cdots, n-1$$

for u a suitable unitary in $\mathrm{Mat}_N(\mathcal{A})$ $\quad \mathcal{A}$ a deformed algebra

This will determine it to a good extent

$2n = 2$ 2-dim sphere S^2

the volume form

$$\mu = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$$

$$x^2 + y^2 + z^2 = 1$$

corresponds to the 2-cycle

$$c = \frac{1}{2}\Big[x \otimes (y \otimes z - z \otimes y) + y \otimes (z \otimes x - x \otimes z) + z \otimes (x \otimes y - y \otimes x)\Big]$$

$$bc = 0$$

Start with $F = \mathbb{C}\langle x, y, z \rangle$ free $*$-algebra

take $e = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \in Mat_2(F)$

Impose

$e = e^* \Rightarrow x, y, z$ self-adjoint

$$e^2 = e \Rightarrow \begin{cases} xy = yx \quad yz = zy \quad zx = xz \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

Thus

$$\mathcal{A} := F \Big/ \begin{smallmatrix} e = e^* \\ e^2 = e \end{smallmatrix} \simeq \mathcal{A}(S^2)$$ polynomial functions on S^2

The component of the Chern-character

$$ch_0(e) := tr\,(e - 1/2) = 0$$

$$ch_1(e) := tr\,(e - 1/2)\, \delta e\, \delta e =$$

$$= \tfrac{1}{2}\, x \otimes (y \otimes z - z \otimes y) + \text{cyc. perm.}$$

In fact in order to identify $C^\infty(S^2)$ one needs also the Dirac operator

D on $\mathcal{H} = L^2(\text{spinors})$

Imposing

$$\pi_D \, ch_1(p) = \text{“}\gamma_5\text{”} \qquad D = \gamma^\mu (\partial_\mu + \dots) \quad \text{Dirac}$$

$$\langle (p - 1/2)[D,p][D,p] \rangle = \text{“}\gamma_5\text{”}$$

a particular [illegible]

$$\pi_D : \delta f \longmapsto [D, f] \in \mathcal{B}(\mathcal{H})$$

shows that $ch_1 = c$ of the volume form μ.

Since μ has support on all of S^2 one concludes that $X = S^2$

Then $\mathcal{A} = C^\infty(S^2)$.

QUANTUM SPHERES

Woronowicz

$$S^3_q \simeq SU_q(2)$$

$$\begin{pmatrix} z_0 & -q z_1^* \\ z_1 & z_0^* \end{pmatrix}$$

$$z_0 z_1^* = q^{-1} z_1^* z_0 \qquad z_0 z_1 = q^{-1} z_1 z_0$$
$$z_1 z_0^* = q^{-1} z_0^* z_1 \qquad z_1^* z_0^* = q^{-1} z_0^* z_1^*$$

$$z_1 z_1^* = z_1^* z_1$$

$$z_0 z_0^* + z_1 z_1^* = 1 \quad , \quad z_0^* z_0 + q^2 z_1 z_1^* = 1$$

I'll use the counit

$$\epsilon : S^3_q \to \mathbb{C}$$

$$\epsilon(z_0) = \epsilon(z_0^*) = 1$$
$$\epsilon(z_1) = \epsilon(z_1^*) = 0$$

A Quantum U(1) - Principal Bundle

$$U(1) = \mathbb{C}[w, w^*] \quad , \quad w^* = w^{-1}$$

A coaction

$$\Delta_R : S^3_q \times U(1) \longrightarrow S^3_q \otimes U(1)$$

$$\Delta_R(w) \cdot \begin{pmatrix} z_0 & -q z_1^* \\ z_1 & z_0^* \end{pmatrix} = \begin{pmatrix} z_0 \otimes w & -q z_1^* \otimes w^* \\ z_1 \otimes w & z_0^* \otimes w^* \end{pmatrix}$$

The base space (the space of coinvariant)

$$\begin{aligned} S^2_q &= S^{3\ U(1)}_q \\ &= \left\{ a \in S^3_q \ : \ \Delta_R(a) = a \otimes 1 \right\} \\ &= \left\{ z_0 z_1^* , \ z_1 z_0^* , \ z_1 z_1^* \right\} \end{aligned}$$

Podles quantum 2-sphere
standard

THE PROJECTION OF CHARGE -1

$$p = |\psi\rangle\langle\psi| \qquad |\psi\rangle = \begin{pmatrix} z_0 \\ q z_1 \end{pmatrix}$$

$$\langle\psi|\psi\rangle = z_0^* z_0 + q^2 z_1^* z_1 = 1$$

$$p^2 = p = p^*$$

$$p = \begin{pmatrix} z_0 z_0^* & q z_0 z_1^* \\ q z_1 z_0^* & q^2 z_1 z_1^* \end{pmatrix} \in \mathrm{Mat}_2(S_q^2)$$

$$\mathrm{tr}\, p = q^2 + (1-q^2)\, z_0 z_0^* = \text{[illegible]}$$

$$\mathrm{ch}_0(p) = (q^2-1)\, z_1^* z_1$$

rank $\quad \epsilon \circ \mathrm{tr}(p) = 1$

charge $\quad \tau^1 \circ \mathrm{tr}(p) = (q^2-1)(1-q^2)^{-1} = -1$

THE PROJECTION OF CHARGE $+1$

$$q = |\varphi\rangle\langle\varphi| \qquad |\varphi\rangle = \begin{pmatrix} z_0^* \\ z_1^* \end{pmatrix}$$

$$\langle\varphi|\varphi\rangle = z_0 z_0^* + z_1 z_1^* = 1$$

$$q^2 = q = q^*$$

$$q = \begin{pmatrix} z_0^* z_0 & z_0^* z_1 \\ z_1^* z_0 & z_1^* z_1 \end{pmatrix} \in \mathrm{Mat}_2(S^2_q)$$

$$\mathrm{tr}(q) = \bar{q}^2 + (1 - q^{-2})\, z_0^* z_0 = \text{[illegible]}$$

rank $\quad \varepsilon \circ \mathrm{tr}(q) = 1$

charge $\quad \tau^1 \circ \mathrm{tr}(q) = (1-q^2)(1-q^2)^{-1} = 1$

A NON TRIVIAL CYCLIC 0-COCYCLE

A basis of the algebra S^2_q as a $\mathbb{C}$-vector space

$$X(m)\,\zeta(n) \qquad m \in \mathbb{Z}\ , \quad n \in \mathbb{N}_0$$

$$X(m) = \begin{cases} (z_0 \bar{z}_1)^m & m \geq 0 \\ (z_1 \bar{z}_0)^{-m} & m < 0 \end{cases}$$

$$\zeta(n) = (z_1 \bar{z}_1)^n \qquad n \geq 0$$

The 0-cocycle : a trace (singular)

$$\tau^1 : S^2_q \to \mathbb{C}$$

$$\tau^1(X(m)\zeta(n)) = \begin{cases} (1-q^{2n})^{-1} & m=0\ ,\ n>0 \\ 0 & m \neq 0 \text{ or } m=n=0 \end{cases}$$

It is in degree zero because of the structure of the (even) cyclic cohomology $HC^*(S^2_q)$. The latter is obtained by acting with the periodicity operator on $HC^0(S^2_q)$

$$HC^0(S^2_q) = \mathbb{C}[\epsilon] \oplus \mathbb{C}[\tau^1] \oplus \operatorname{Ker} S$$

Instantons on the Euclidean sphere S_q^4

$S_q^4 =$ " $SO_q(5)/SO_q(4)$ "

The generators of the algebra S_q^4 are

$$x^i = (x_2^*, x_1^*, x_0, x_1, x_2),$$

with $(x_0)^* = x_0$; and commutation relations

$$\begin{aligned}
&x_i x_j = q x_j x_i, \quad i < j; \qquad x_i^* x_j = q x_j x_i^*, \quad i \neq j, \\
&[x_1, x_1^*] = (1 - q^{-2}) x_0^2, \\
&[x_2, x_2^*] = (1 - q^{-2}) \left(q x_1 x_1^* + x_0^2 + x_1^* x_1\right)
\end{aligned}$$

The element

$$r^2 = q^3 x_2 x_2^* + q x_1 x_1^* + x_0^2 + x_1^* x_1 + x_2^* x_2$$

is central and can be diagonalized.

Proposition
The element $e \in \mathrm{Mat}_4(S_q^4)$ given by

$$e = \frac{1}{2} \begin{pmatrix} 1 + q^{-2} x_0 & q^{-1} x_1 & x_2 & 0 \\ q^{-1} x_1^* & 1 - q^{-1} x_0 & 0 & x_2 \\ x_2^* & 0 & 1 - q^{-1} x_0 & -x_1 \\ 0 & x_2^* & -x_1^* & 1 + x_0 \end{pmatrix}$$

which is hermitian by construction, is also idempotent, $e^2 = e$, if and only if all the relations which define S_q^4 are satisfied.

$$ch_0(e) = \frac{1}{2}(1 - q^{-1})^2 x_0$$

One needs two traces on the algebra

$\tau^0 \longrightarrow$ rank

$\tau^1 \longrightarrow$ the topological number

is a singular trace

BPST
ADHM

The Instanton

$$\begin{array}{c} S^7 \\ \pi \downarrow SU(2) \\ S^4 \end{array} \qquad \text{Hopf bundle}$$

$$S^7 = \{(a,b) \in \mathbb{H}^2,\ |a|^2 + |b|^2 = 1\}$$

$$SU(2) \simeq Sp(1) = \{w \in \mathbb{H},\ |w|^2 = 1\}$$

$$S^7 \times Sp(1) \to S^7 \qquad (a,b)w = (aw, bw)$$

$$\pi(a,b) = (x_\mu,\ \mu = 0, \dots, 4)$$

$$x_0 = |a|^2 - |b|^2$$

$$x_4 = a\bar{b} + b\bar{a}$$

$$\xi = x_1 i + x_2 j + x_3 k = a\bar{b} - b\bar{a}$$

$$\sum_{\mu=0}^{4} (x_\mu)^2 = 1$$

The $\mathbb{H}$-valued invariant functions on S^7

$$|a|^2 = \frac{1}{2}(1 + x_0) \qquad |b|^2 = \frac{1}{2}(1 - x_0)$$

$$a\bar{b} = \frac{1}{2} x_4 + \xi$$

The instanton bundle

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \langle\psi|\psi\rangle = 1$$

$$e = |\psi\rangle\langle\psi| \in Mat_2(\mathcal{A}) \qquad e^2 = e = e^*$$

$$\mathcal{A} = C^\infty(S^4, \mathbb{H})$$

$$e = \begin{pmatrix} |a|^2 & a\bar{b} \\ b\bar{a} & |b|^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} I + x_0 & x_4 + \xi \\ x_4 - \xi & I - x_0 \end{pmatrix}.$$

The sections of a vector bundle $E \to M$

$\Gamma = e\mathcal{A}^2$

$$\Gamma(S^4, E) \longleftrightarrow C^\infty_{Sp(1)}(S^7, \mathbb{H})$$

$$\sigma = e\begin{pmatrix} f \\ g \end{pmatrix} \longleftrightarrow \varphi_\sigma = \langle\psi|\begin{pmatrix} f \\ g \end{pmatrix} = \bar{a}f + \bar{b}g$$

$$f, g \in \mathcal{A}$$

$$\varphi_\sigma((a,b)\cdot w) = w^{-1}\varphi_\sigma(a,b)$$

$tr(e) = 2$ (over $\mathbb{C}$)

the rank of the bundle

the canonical connection

$$\nabla = e \circ d : \Gamma \longrightarrow \Gamma \otimes_{\mathcal{A}} \Omega^1$$

its curvature

$$\nabla^2 = e(de)^2$$

$$= |\psi\rangle\langle\psi|d\psi\rangle^2\langle\psi| + |\psi\rangle\langle d\psi|d\psi\rangle\langle\psi|$$

The Chern classes

$$c_1(e) = +\frac{1}{2\pi i} \operatorname{tr} e(de)^2 = \dots = 0$$

$$c_2(e) = +\frac{1}{8\pi^2} \operatorname{tr} e(de)^4 = +\frac{3}{8\pi^2} d(\mathrm{vol}(S^4))$$

$$\int_{S^4} c_2(e) = +1$$

The connection 1-form

$$A = \langle\psi|d\psi\rangle = \bar{a}\, da + \bar{b}\, db$$

and is anti self dual

$$* \nabla^2 = - \nabla^2$$

Hodge map

Toward quantum spheres

$$e = \frac{1}{2}\begin{pmatrix} (1+z)\mathbb{I} & \mathcal{Q} \\ \mathcal{Q}^* & (1-z)\mathbb{I} \end{pmatrix}$$

$$\mathcal{Q} = \begin{pmatrix} \alpha & \beta \\ -q\beta^* & \alpha^* \end{pmatrix}$$

α, β complex gen.

$z = z^*$

The requirement that $e^2 = e$ idempotent

$$z\alpha = \alpha z \qquad z\beta = \beta z$$

$$\beta\alpha = \bar{q}\alpha\beta \qquad \beta^*\alpha = q\alpha\beta^*$$

$$\beta\beta^* = \beta^*\beta$$

$$\alpha\alpha^* + \beta\beta^* + z^2 = 1$$

$$\alpha^*\alpha + |q|^2\beta^*\beta + z^2 = 1$$

One can restrict the deformation parameter $q \in \mathbb{C}$ so that $|q| \in (0, 1]$

The Connes-Chern classes

$$ch_j(e) = \langle (e - 1/2)\, de^{2j} \rangle$$

$$e^2 = e$$

$$bch_{j+1}(e) = Bch_j(e)$$

For the previous idempotent

- $ch_0(e) = \langle e - 1/2 \rangle = 0$

  $\mathrm{rank} = 2$

- $ch_1(e) \sim (1 - |q|^2)\,(\cdots\cdots)$

- $ch_2(e) \sim (1 - |q|^4)\,(\quad)$

  $+$

  $(1 - |q|^2)\,(\quad)$

  $+$

  q-deformation of the classical term

Dabrowski, L., Masuda

$q \in \mathbb{R}$ $\qquad\qquad$ S^4_q

a suspension of $\quad S^3_q \simeq SU_q(2)$

$z\alpha = \alpha z \qquad\qquad z\beta = \beta z$

$\beta\alpha = q\alpha\beta \qquad\qquad \beta^*\alpha = q\alpha\beta^*$

$\beta\beta^* = \beta^*\beta$

$\alpha\alpha^* + \beta\beta^* + z^2 = \mathbb{1}$

$\alpha^*\alpha + q^2\beta^*\beta + z^2 = \mathbb{1}$

$\beta = z_1 \qquad \alpha = z_0$

S^4_θ $\Big\}$ S^{2n}_θ Connes Dubois-Violette

take $q = e^{2\pi i\theta} = \lambda$

$ch_1(e) = 0$

$ch_2(e) =$ λ-deformation of the classical term

- it is λ- [illegible]
- $b\, ch_2(e) = 0$

the associated Hochschild class is a volume form

$C(S^4_\theta)$ is the $*$-algebra generated by $z, \alpha, \beta, \alpha^*, \beta^*$ $\quad z^* = z$

$$z\alpha = \alpha z \qquad z\beta = \beta z$$
$$\alpha\beta = \lambda\beta\alpha \qquad \alpha\beta^* = \bar{\lambda}\beta^*\alpha \qquad \lambda = e^{2\pi i\theta}$$
$$\alpha\alpha^* = \alpha^*\alpha \qquad \beta\beta^* = \beta^*\beta$$

$$\alpha\alpha^* + \beta\beta^* + z^2 = 1$$

The θ-deformed spheres are noncommutative homogeneous spaces, i.e. subalgebra of coinvariants of suitable Hopf algebras which can be regarded as quantized symmetry groups

$$S^{N-1}_\theta = \text{"} SO_\theta(N) / SO_\theta(N-1) \text{"}$$

$$\delta : S^{N-1}_\theta \longrightarrow SO_\theta(N) \otimes S^N_\theta$$

$$\delta(\text{radius}) = 1 \otimes \text{radius}$$

The groups $SO_\theta(N)$ are compact matrix quantum groups

THE NC GEOMETRY OF S^4_θ

a spectral triple $\mathcal{A}_\theta = C^\infty(S^4_\theta)$, $\mathcal{H}$, D

we shall deform the canonical triple of S^4

$\mathcal{A} = C^\infty(S^4)$

$\mathcal{H} = L^2(S^4, S)$ spinors

$D = \gamma^\mu(\partial_\mu + \omega_\mu)$ Dirac operator

The crucial operator equation for D will be

$$\langle (e - 1/2)[D, e]^4 \rangle = \gamma_5$$

We shall keep $\mathcal{H}$ and D the same while changing ONLY the algebra and the way it acts on $\mathcal{H}$

AN ISOSPECTRAL DEFORMATION

- DEFORMING THE ALGEBRA & ITS REPRESENTATION

A spectral decomposition

$T^2 \subset \mathrm{Isom}(S^4)$

decompose $f \in C^\infty(S^4)$ $\qquad f = \sum_n f_n \quad n \in \mathbb{Z}^2$

$$(e^{2\pi i \varphi_1}, e^{2\pi i \varphi_2}) \cdot f_n = e^{2\pi i (n_1 \varphi_1 + n_2 \varphi_2)} f_n$$

$$n = (n_1, n_2)$$

a $*_\theta$ product

$$f_n *_\theta g_m = e^{\pi i \theta (n_2 m_1 - n_1 m_2)} f_n \, g_m$$

The generators (p_1, p_2) of $U(\varphi)$ $\varphi \in T^2$

the unitary reps on $\mathcal{H} = L^2(M, S)$

The left twist

$$\ell(f) = \Sigma_n f_n \, \lambda^{n_2 p_1} \qquad \lambda = e^{2\pi i \theta}$$

$$\ell(f * g) = \ell(f)\, \ell(g)$$

The algebra $\ell(C^\infty(S^4))$ acts on $\mathcal{H}$

and coincides with $\mathcal{A}_\theta = C^\infty(S^4_\theta)$

Parametrize the algebra generators
α, β, z

$$\alpha = \tfrac{1}{2} u \cos\varphi \cos\psi \qquad \beta = \tfrac{1}{2} v \sin\varphi \cos\psi$$

$$z = \tfrac{1}{2} \sin\psi$$

$$0 \leq \varphi \leq \pi/2 \qquad -\pi/2 \leq \psi \leq \pi/2$$

$u, v \in C^\infty(T^2_\theta)$ the algebra of the nc Torus.

$$uv = \lambda v u \qquad uu^* = u^* u = 1 = vv^* = v^* v$$

For $\theta = 0$ write
the metric, the volume form and D
using the variable φ, ψ, u, v

then ...

---

fn. T^2 acts by rotation on S^4_θ
$u \to e^{2\pi i \varphi_1} u \qquad v \to e^{2\pi i \varphi_2} v$

The nc Dirac operator is

$$D = \frac{1}{\cos\varphi\cos\psi}\,\delta_1\gamma_1 + \frac{1}{\sin\varphi\cos\psi}\,\delta_2\gamma_2$$
$$+ \frac{\sqrt{-1}}{\cos\psi}\left(\frac{\partial}{\partial\varphi} + \frac{1}{2}\cot g\,\varphi - \frac{1}{2}\,tg\,\varphi\right)\gamma_3$$
$$+ \sqrt{-1}\left(\frac{\partial}{\partial\psi} - \frac{3}{2}\,tg\,\psi\right)\gamma_4$$

$\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are [illegible]

δ_1, δ_2 the n.c.t. derivations

$$\delta_1(u) = u \qquad \delta_1(v) = 0$$
$$\delta_2(u) = 0 \qquad \delta_2(v) = v$$

One finds that

$$\langle (e - 1/2)[D, e]^4 \rangle = \gamma_5$$

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$$

- The spectral triple $(C^\infty(S^4_\theta), \mathcal{H}, D)$ fulfills all axioms of nc g.
- If $e \in C^\infty(S^4_\theta, M_4(\mathbb{C}))$ is the canonical idempotent the Dirac operator D fulfills

$$\langle (e - \tfrac{1}{2})[D, e]^4 \rangle = \gamma_5$$

∎

The metric is the 'round one'.
One can extend this theorem to any metric on S^4 invariant under rotation of u, v and whose volume form $\sqrt{g}\,dx$ is the one of the round one.

More generally

Let M be a compact spin Riemannian mfld whose isometry group has rank $k \geq 2$.
Then M admits a natural 1-par family of isospectral deformation to ncg M_θ

MORE EXAMPLES FROM THE NC TORUS

$\mathcal{A}_\theta = C(\mathbb{T}^n_\theta)$ the algebra of the nct $\mathbb{T}^n_\theta$

generated by n unitary elements

$U^\mu \quad \mu = 1, \dots, n$

$$U^\nu U^\mu = \lambda^{\nu\mu} U^\mu U^\nu$$

$$\lambda^{\mu\nu} = -\lambda^{\nu\mu} \qquad |\lambda^{\mu\nu}| = 1$$

$n = 2 \qquad \lambda = e^{2\pi i \theta} \qquad U^2 U^1 = e^{2\pi i \theta} U^1 U^2$

An action of commutative $\mathbb{T}^n$ on nc $\mathbb{T}^n_\theta$

$$\mathbb{T}^n \ni s \longrightarrow \tau_s \in \mathrm{Aut}(\mathcal{A}_\theta)$$

$$\tau_s(U^\mu) = e^{2\pi i s_\mu} U^\mu$$

$$s = (s_\mu ;\ \mu = 1, \dots, n)$$

M a mfld with a $\mathbb{T}^m$ action

$$\mathbb{T}^m \ni s \longmapsto \sigma_s \in \mathrm{Aut}\,(C(M))$$

Take $\quad C(M \times \mathbb{T}^n_\theta) \simeq C(M) \hat{\otimes} A_\theta$

there is a diagonal action

$$\sigma \times \tau^{-1} : \mathbb{T}^n \longrightarrow \mathrm{Aut}\,(C(M \times \mathbb{T}^n_\theta))$$

The deformed algebra is the invariant subalgebra

$$C(M_\theta) = C(M \times \mathbb{T}^n_\theta)^{\sigma \times \tau^{-1}}$$

SMOOTH STRUCTURE

$C^\infty(\mathbb{T}^n_\theta)$ the completion of $C(\mathbb{T}^n_\theta)$ in the l. convex topology generated by seminorms

$$|a|_r = \sup_{r_1+\dots+r_n \leq r} \| X_1^{r_1} \cdots X_n^{r_n}(a) \|$$

$\| \cdot \|$ C^*-norm

X_μ the infinitesimal generators of the action of $\mathbb{T}^n$ on $\mathbb{T}^n_\theta$

$$X_\mu (U^\nu) = 2\pi i \, \delta_\mu{}^\nu \, U^\nu$$

$C^\infty_c(M_\theta)$ is the fixed point subalgebra of the diagonal action of $\mathbb{T}^n$ on the completed top. product

$$C^\infty_c(M) \hat{\otimes} C^\infty(\mathbb{T}^n_\theta)$$

$$C^\infty_c(M_\theta) = C^\infty(M) \hat{\otimes} C^\infty(\mathbb{T}^n_\theta)^{\sigma \times \tau^{-1}}$$

DIFFERENTIAL FORMS

$$\Omega(M_\theta) = \Omega(M) \hat{\otimes} C^\infty(\mathbb{T}^n_\theta)^{\sigma \times \tau^{-1}}$$

$$d = d \otimes \mathrm{Id}$$

HODGE THEORY

$\omega \longmapsto * \omega$ the Hodge operator on M

w.r. to a σ invariant metric on M

it can be extended to an endomorphism $*_\theta$ of $\Omega(M_\theta)$ as a bimodule over $C^\infty(M_\theta)$

$$*_\theta : \Omega^p(M_\theta) \to \Omega^{n-p}(M_\theta)$$

$$*_\theta = * \otimes \mathrm{Id}$$

$M = \mathbb{R}^{2n}$

$z^{\mu}_{(0)}$ $\quad \mu = 1, \dots, n$ $\quad$ generator of $C(\mathbb{R}^{2n})$

$C(\mathbb{R}^{2n}_{\theta})$ is generated by

$$z^{\mu} = z^{\mu}_{(0)} \otimes U^{\mu} \qquad \mu = 1, \dots, n$$

$$z^{\mu} z^{\nu} = \lambda^{\mu\nu} z^{\nu} z^{\mu}$$
$$\bar{z}^{\mu} \bar{z}^{\nu} = \lambda^{\mu\nu} \bar{z}^{\nu} \bar{z}^{\mu}$$
$$\bar{z}^{\mu} z^{\nu} = \lambda^{\nu\mu} z^{\nu} \bar{z}^{\mu}$$

$\sum_{\mu=1}^{n} z^{\mu} \bar{z}^{\mu}$ is central

$$C(S^{2n-1}_{\theta}) = C(\mathbb{R}^{2n}_{\theta}) \Big/ \sum_{\mu} z^{\mu} \bar{z}^{\mu} - 1$$

A unitary $\qquad u \in \mathrm{Mat}_{2^{n-1}}(C(S^{2n-1}_{\theta}))$

$$ch_{k+1/2}(u) = 0 \qquad k = 0, 1, \dots, n-2$$

$$u = \sum_{\mu=1}^{n} \bar{\sigma}^{\mu} z^{\mu} + \sigma^{\mu} \bar{z}^{\mu}$$

For $C(\mathbb{R}^{2n+1}_\theta)$ an additional central element $x = \bar{x}$

$$x z^\mu = z^\mu x \quad ; \quad x \bar{z}^\mu = \bar{z}^\mu x \qquad \mu = 1, 2, \cdots, n$$

$$C(S^{2n}_\theta) = C(\mathbb{R}^{2n+1}_\theta) \Big/ \sum_\mu z^\mu \bar{z}^\mu + x^2 - 1$$

A projection $\quad e \in \mathrm{Mat}_{2^n}(C(S^{2n}_\theta))$

$$ch_k(e) = 0 \qquad k = 0, 1, \dots, n-1$$

$$e = \frac{1}{2}\Big(1 + \sum_{\mu=1}^{n} (\bar{\Gamma}^\mu z^\mu + \Gamma^\mu \bar{z}^\mu) + \gamma x\Big)$$

$$\Gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \bar{\Gamma}^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix}$$

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

THM. Let S^{2n} be endowed with it usual (round) metric; let $*_\theta$ be the corresponding operator on $\Omega(S^{2n}_\theta)$;

Then the 'defining' projection e is self-dual, i.e.

$$*_\theta e(de)^n = i^n e(de)^n$$

The classical limit $(\theta = 0)$ of this equation describes an instanton for a conformally invariant generalization of the classical Y-M action on S^{2n}

On S^4 we have the usual YM action

Let g any $\mathbb{T}^n$-invariant metric on S^{2n} whose volume form is the same as for the round metric ;

D its Dirac operator on $\mathcal{H} = L^2(S^{2n}, \text{Spinor})$

Then

- the spectral triple $(C(S^{2n}_\theta), \mathcal{H}, D)$ fulfills all axioms of n.c. g.

- if e is the defining projection ; it fulfills

$$\langle (e - \tfrac{1}{2})[D, e]^{2n} \rangle = \gamma_5$$

The topological charge of e can be computed by an index

$$K_0(\quad) \ni [e] \longmapsto \text{Index } D_e^+ \in \mathbb{Z}$$

$$D_e^+ = e D^+ e \qquad D^+ = D\left(\frac{1+\gamma}{2}\right)$$

As a consequence of the fact that only the top component of the Chern character is $\neq 0$, i.e.

$$ch_j(e) = 0 \qquad j = 0, 1, \dots, n$$

the local expression for the index is simply

$$\operatorname{Index} D_e^+ = (-1)^n \int \gamma \, (e - \tfrac{1}{2}) [D, e]^{2n} |D|^{-2n}.$$

For the defining projection

$$\operatorname{Index} D_e^+ = (-1)^n \int \gamma \cdot \gamma \; |D|^{-2n}$$

$$= (-1)^n$$

Monopoles over projective spaces

$$\begin{tikzcd} S^{2n+1} \arrow[d, "U(1)"] \arrow[r, dashed] & S^{2n+1}_{\Theta} \arrow[d, "U(1)"] \\ \mathbb{C}P^n & \mathbb{C}P^n_{\Theta'} \end{tikzcd}$$

$p = |\psi\rangle\langle\psi|$ $\qquad |\psi\rangle = \begin{pmatrix} z^1 \\ \vdots \\ z^n \end{pmatrix}$, $\langle\psi|\psi\rangle = 1$

p_{ij} generators for $\mathbb{C}P^n_{\Theta'}$

$p^2 = p^* = p$ a line bundle over $\mathbb{C}P^n_{\Theta'}$

$bch_1 = 0$

in particular $n = 2$

$$\begin{tikzcd} S^5_{\Theta} \arrow[d] \\ \mathbb{C}P^2_{\theta} \end{tikzcd}$$

$\lambda = \lambda^{01}\lambda^{12}\lambda^{20}$

$\lambda = e^{2\pi i\Theta}$

Twistor spaces & principal bdl.s

$$\begin{tikzcd} S^7 \arrow[r, "U(1)"] \arrow[dr, "SU(2)"'] & \mathbb{C}P^3 \arrow[d, "S^2"] \\ & S^4 \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} S^7_\theta \arrow[r, "U(1)"] \arrow[dr, "SU(2)"'] & \mathbb{C}P^3_\theta \arrow[d, "S^2"] \\ & S^4_\theta \end{tikzcd}$$

a suitable coaction of

$C^\infty(SU(2))$ on $C^\infty(S^7_\theta)$

An analogous construction on $\mathbb{C}P^2_\theta$

$$\begin{tikzcd} \mathbb{F}^6 = SU(3)/U(1)\times U(1) \arrow[d, "S^2"] & & \mathbb{F}^6_\theta \arrow[d, "S^2"] \\ \mathbb{C}P^2 \quad SU(3)/U(2) \arrow[rr, phantom, "\Longrightarrow"] & & \mathbb{C}P^2_\theta \end{tikzcd}$$

the $\mathbb{T}^2$ action on $\mathbb{C}P^2$ is lifted to the Kähler mfld $\mathbb{F}^6$

- A homological construction of nc 4D geometries

  $(\mathcal{A}_\theta, \mathcal{H}, D, \gamma)$

  $$ch_k(e) = 0 \qquad k = 0, 1$$

  $$e^2 = e = e^*$$

  $$\pi_D\, ch_2(e) = \gamma$$

  $$\pi_D(a_0 \delta a_1 \cdots \delta a_n) = a_0 [D, a_1] \cdots [D, a_n] \in \mathcal{B}(\mathcal{H})$$

- An isospectral deformation:
  all spectral data, including the dimension ([illegible]) are unchanged

- A nc instanton bundle

  $E = e\,\mathcal{A}_\theta^4$ the module of sections

  $\nabla = e \circ d$ a preferred connection

The starting ingredients for Yang-Mills theory on S^4_θ

- The Yang-Mills action

$$YM(\nabla) = ⨍ F^2 \, ds^4$$

$$F = \nabla^2 \quad \text{urvature}$$

$$ds = |D|^{-1}$$

with a strictly positive lower bound given by

$$\mathrm{Top}(e) = ⨍ \gamma \left(e - \tfrac{1}{2}\right) [D, e]^4 \, ds^4$$

The defining projection is

a. self-dual $\quad * \nabla^2 = - \nabla^2$

and $\quad \mathrm{Top}(e) = 1$

The basic quartic equation for D

$$\langle (e - 1/2)[D, e]^4 \rangle = \gamma_5$$

does not fix the metric but only the volume form $\sqrt{g}\, dx$

. any metric with the fixed volume form yields a solution

volume preserving variations of the metric

different solution could be compared using the SPECTRAL ACTION

$$\mathrm{tr}\,(\chi(D/\Lambda)) = \Lambda^4 \int ds^4 + \Lambda^2 \int R + \ldots$$

which could select the 'CORRECT ONE'

(modulo Diff, of course) .

CL $\qquad\qquad\qquad\qquad$ CD-B

and only one .... (only a class of ...)

a spin non commutative geometry

$(\mathcal{A}, \mathcal{H}, D, \gamma)$ $\qquad\qquad$ $\mathcal{A} = C^\infty(S^4_\theta)$

solution of

$$\mathrm{ch}_k(e) = 0 \qquad k = 0, 1$$

$$\pi_D(\mathrm{ch}_2(e)) = \gamma$$

$e = e^2 = e^*$ a projection in $M_4(\mathcal{A})$

$$\pi_D(a_0\, da_1 \cdots da_n) = a_0 [D, a_1] \cdots [D, a_n]$$

$$\mathrm{ch}_j(e) = \langle (e - 1/2)\, de^{2j} \rangle \in \mathcal{A} \otimes \underbrace{\bar{\mathcal{A}} \otimes \cdots}_{2n}$$

A partial trace $\qquad \bar{\mathcal{A}} = \mathcal{A} / \mathbb{C}\mathbb{1}$

a cycle in the (b, B) bicomplex of cyclic homology

$$B\,\mathrm{ch}_j(e) = b\,\mathrm{ch}_{j+1}(e)$$

$ch_0(e) = \langle (e - 1/2) \rangle = 0$ $\qquad$ rank$(e) = 2$

$ch_1(e) = 0 \quad \Rightarrow$

$b\, ch_2(e) = 0$

The cycle $ch_2(e)$ can be used as a volume form

$ch_2(e)$ is a Hochschild cycle

	Classical	C.L	DLM S BG	BCT LM (emb. ?)
ch_0	0	0	0	$\neq 0$
ch_1	0	0	$\neq 0$	
ch_2	1	1		

Connes Dubois-Violette S^{2m}

$ch_k = 0 \qquad k = 0, 1, \dots, m-1$

ch_m is the volume form.

The classical points

1-dim IR , characters

$$\varphi : \mathcal{A} \longrightarrow \mathbb{C} \quad , \quad \varphi(ab) = \varphi(a)\varphi(b)$$

BCT	$\bullet$	
HS LM	S^1	
S BG	$S^1 \times \mathbb{Z}_2$	
DL DLM	S^2	
CL C D-V	$S^2 \times \mathbb{Z}_2$	the 2-spheres intersect at the poles