Topology of Incompressible Fermionic Quantum Fluids

R. Shankar

The Institute of Mathematical Sciences, Chennai

IMSc@50
One day (14 March 2012) Meeting on
Topology and Differential geometry in quantum physics

Outline

The Quantum Hall Effect and Topological order

Topological Insulators

Kitaev-Hubbard Model

Outline

The Quantum Hall Effect and Topological order

Topological Insulators

Kitaev-Hubbard Model

IQHE

VOLUME 45, NUMBER 6 PHYSICAL REVIEW LETTERS 11 AUGUST 1980

New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance

K. v. Klitzing
Physikalisches Institut der Universität Würzburg, D-8700 Würzburg, Federal Republic of Germany, and Hochfeld-Magnetlabor des Max-Planck-Instituts für Festkörperforschung, F-38042 Grenoble, France

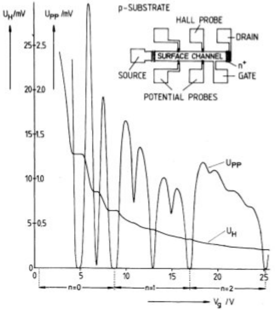
and

G. Dorda
Forschungslaboratorien der Siemens AG, D-8000 München, Federal Republic of Germany

and

M. Pepper
Cavendish Laboratory, Cambridge CB3 0HE, United Kingdom

(Received 30 May 1980)



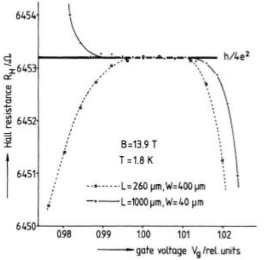


FIG. 3. Hall resistance R_H for two samples with different geometry in a gate-voltage region V_g where the $n=0$ Landau level is fully occupied. The recommended value $h/4e^2$ is given as 6453.204 Ω .

FQHE

VOLUME 48, NUMBER 22 PHYSICAL REVIEW LETTERS 31 MAY 1982

Two-Dimensional Magnetotransport in the Extreme Quantum Limit

D. C. Tsui,[a],[b] H. L. Stormer,[a] and A. C. Gossard
Bell Laboratories, Murray Hill, New Jersey 07974
(Received 5 March 1982)

A quantized Hall plateau of $\rho_{xy} = 3h/e^2$, accompanied by a minimum in ρ_{xx}, was observed at $T < 5$ K in magnetotransport of high-mobility, two-dimensional electrons, when the lowest-energy, spin-polarized Landau level is $\frac{1}{3}$ filled. The formation of a Wigner solid or charge-density-wave state with triangular symmetry is suggested as a possible explanation.

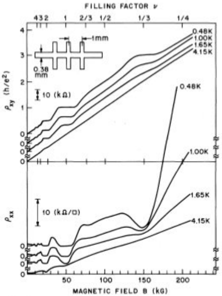


FIG. 1. ρ_{xy} and ρ_{xx} vs B, taken from a GaAs-$Al_{0.3}Ga_{0.7}As$ sample with $n = 1.23\times10^{11}/cm^2$, $\mu = 90\,000$ cm^2/V sec, using $I = 1\ \mu$A. The Landau level filling factor is defined by $\nu = nh/eB$.

σ_H=Chern Invariant

VOLUME 49, NUMBER 6 PHYSICAL REVIEW LETTERS 9 AUGUST 1982

Quantized Hall Conductance in a Two-Dimensional Periodic Potential

D. J. Thouless, M. Kohmoto,[a] M. P. Nightingale, and M. den Nijs
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 30 April 1982)

The Hall conductance of a two-dimensional electron gas has been studied in a uniform magnetic field and a periodic substrate potential U. The Kubo formula is written in a form that makes apparent the quantization when the Fermi energy lies in a gap. Explicit expressions have been obtained for the Hall conductance for both large and small $U/\hbar\omega_c$.

PACS numbers: 72.15.Gd, 72.20. Mg, 73.90.+b

$$\sigma_{\mathrm{H}} = \frac{ie^2}{2\pi h}\sum\int d^2k\int d^2r\left(\frac{\partial u^*}{\partial k_1}\frac{\partial u}{\partial k_2} - \frac{\partial u^*}{\partial k_2}\frac{\partial u}{\partial k_1}\right)$$

$$= \frac{ie^2}{4\pi h}\sum\oint dk_j\int d^2r\left(u^*\frac{\partial u}{\partial k_j} - \frac{\partial u^*}{\partial k_j}u\right), \qquad (5)$$

σ_H=No. of edge channels

VOLUME 71, NUMBER 22 PHYSICAL REVIEW LETTERS 29 NOVEMBER 1993

Chern Number and Edge States in the Integer Quantum Hall Effect

Yasuhiro Hatsugai
Department of Physics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139
and Institute for Solid State Physics, University of Tokyo, 7-22-1 Roppongi Minato-ku, Tokyo 106, Japan

(Received 12 July 1993)

We consider the integer quantum Hall effect on a square lattice in a uniform rational magnetic field. The relation between two different interpretations of the Hall conductance as topological invariants is clarified. One is the Thouless–Kohmoto–Nightingale–den Nijs (TKNN) integer in the infinite system and the other is a winding number of the edge state. In the TKNN form of the Hall conductance, a phase of the Bloch wave function defines U(1) vortices on the magnetic Brillouin zone and the total vorticity gives σ_{xy}. We find that these vortices are given by the edge states when they are degenerate with the bulk states.

PACS numbers: 73.40.Hm, 02.40.–k

and that at the band bottom is $-I(C_{j-1})$. Therefore the Hall conductance of the filled jth band is

$$\sigma_{xy}^{j,\text{bulk}} = -\frac{e^2}{h}[\, I(C_j) - I(C_{j-1})\,] = \sigma_{xy}^{j,\text{edge}}, \qquad (15)$$

Non-interacting fermions on a lattice

$$H = \int \frac{d^d k}{(2\pi)^d} C^\dagger(k)_\alpha \left(h(k) - \epsilon_F\right)_{\alpha\beta} C(k)_\beta$$

$\alpha, \beta = 1, \cdots, N_B$

$$h(k)_{\alpha\beta} = U(k)_{\alpha\gamma}\epsilon(k)_\gamma U^\dagger(k)_{\gamma\beta}$$

The ground state depends only on $U(k)$. Many hamiltonians will share the ground state with H. In particular if the Fermi level lies in a gap,

$$\begin{aligned} Q &= \int \frac{d^d k}{(2\pi)^d} C^\dagger(k) q(k) C(k) \\ q(k)_{\alpha\beta} &= U(k)_{\alpha\gamma}\Lambda_{\gamma\gamma'} U^\dagger(k)_{\gamma'\beta} \\ \Lambda_{\gamma\gamma'} &= \mathrm{sgn}(\epsilon(k)_\gamma - \epsilon_F)\delta_{\gamma\gamma'} \end{aligned}$$

$d = 2,\ N_B = 2$

$$
\begin{aligned}
h(k) &= U(k)\begin{pmatrix} \epsilon_1(k) & 0 \\ 0 & \epsilon_2(k) \end{pmatrix} U^\dagger(k) \\
q(k) &= U(k)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U^\dagger(k) \\
&= \hat{n}(k)\cdot\vec{\tau}
\end{aligned}
$$

Thus $q(k)$ defines a map from the 2-torus to the 2-sphere. This map breaks up into topological sectors characterised by the integer valued Chern invariant.
TKNN work showed that a physical property shared by all ground states in a topological sector is,

$$\sigma_H = \frac{e^2}{h} \times \nu \quad (\nu = \text{Chern Invariant})$$

$d = 2$, N_B bands, N_F occupied

$$
\begin{aligned}
q(k) &= U(k)\Lambda U^\dagger(k) \in \frac{U(N_B)}{U(N_F)\times U(N_B - N_F)} \\
\nu &= \frac{i}{8\pi}\int d^2k\, \epsilon_{ij}\mathrm{tr}\left(q\left[\partial_i q, \partial_j q\right]\right)
\end{aligned}
$$

The Haldane model

VOLUME 61, NUMBER 18 PHYSICAL REVIEW LETTERS 31 OCTOBER 1988

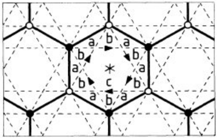
Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the "Parity Anomaly"

F. D. M. Haldane

Department of Physics, University of California, San Diego, La Jolla, California 92093

(Received 16 September 1987)

A two-dimensional condensed-matter lattice model is presented which exhibits a nonzero quantization of the Hall conductance σ^{xy} in the *absence* of an external magnetic field. Massless fermions *without spectral doubling* occur at critical values of the model parameters, and exhibit the so-called "parity anomaly" of (2+1)-dimensional field theories.



$$\mathbf{H}(\mathbf{k}) = 2t_2 \cos\phi \left[\textstyle\sum_i \cos(\mathbf{k}\cdot\mathbf{b}_i)\right] \mathbf{I} + t_1 \left[\textstyle\sum_i [\cos(\mathbf{k}\cdot\mathbf{a}_i)\boldsymbol{\sigma}^1 + \sin(\mathbf{k}\cdot\mathbf{a}_i)\boldsymbol{\sigma}^2]\right] + \left[M - 2t_2\sin\phi\left(\textstyle\sum_i \sin(\mathbf{k}\cdot\mathbf{b}_i)\right)\right]\boldsymbol{\sigma}^3, \qquad (1)$$

Non-zero Chern number without a magnetic field

$$
\begin{aligned}
h(k) &= \alpha^x p_x(k) + \alpha^y p_y(k) + \beta M(k) \\
p_x(k) &= t(1 + \cos k_1 + \cos k_2) \\
p_y(k) &= t(\sin k_1 - \sin k_2) \\
M(k) &= M + \Delta\left(\sin k_1 + \sin k_2 + \sin(-k_1 - k_2)\right)
\end{aligned}
$$

- $M >> \Delta,\ \nu = 0$, $M << \Delta,\ \nu = \pm 1$. Topological transition from $\nu = 0$ to $\nu = \pm 1$ phase at $M = 3\frac{\sqrt{3}}{2}$.
- Magnetic field not necessary for non-zero Chern number. Other time reversal symmetry breaking terms can also induce it.

Degeneracy on a torus

PHYSICAL REVIEW LETTERS

VOLUME 55 | 11 NOVEMBER 1985 | NUMBER 20

Many-Particle Translational Symmetries of Two-Dimensional Electrons at Rational Landau-Level Filling

F. D. M. Haldane[a]

Department of Physics, University of Southern California, Los Angeles, California 90089

(Received 11 February 1985)

In contrast to previous treatments, a new analysis of two-dimensional many-electron systems subject to periodic boundary conditions in a magnetic field leads to a fully two-dimensional structure of the quantum numbers at rational Landau-level filling. The structure of the new symmetry analysis has an intrinsically many-particle character. Full agreement between numerical studies of quantized-Hall-effect systems in periodic and spherical geometries is achieved, and the problem of ground-state degeneracy is clarified.

It should be emphasized that this q-fold c.m. degeneracy is a purely group-theoretical consequence of the imposition of PBC's on a translationally invariant system, and *quite without physical significance.* It is a degeneracy common to *every* eigenvalue of the Hamiltonian belonging to a given subspace $\mathscr{H}(\theta_1, \theta_2)$, and related to the degeneracy between subspaces. It is present *independent of the physical nature of the ground state of* H^{rel}, whether it is of the fluid type and exhibits the QHE, or a solid type and does not.

Magnetic tranlations

$$T_a T_b T_a^{-1} T_b^{-1} |\psi\rangle_N = e^{i2\pi\nu}$$

- If $\nu = \frac{p}{q}$, then lowest unitary irreducible representation is q-dimensional.
- If system is translationally invariant, then all eigenstates are atleast q-fold degenerate

Chern invariant for interacting systems

PHYSICAL REVIEW B VOLUME 31, NUMBER 6 15 MARCH 1985

Quantized Hall conductance as a topological invariant

Qian Niu, D. J. Thouless,* and Yong-Shi Wu†
Department of Physics FM-15, University of Washington, Seattle, Washington 98195

(Received 21 September 1984)

Whenever the Fermi level lies in a gap (or mobility gap) the bulk Hall conductance can be expressed in a topologically invariant form showing the quantization explicitly. The new formulation generalizes the earlier result by Thouless, Kohmoto, Nightingale, and den Nijs to the situation where many-body interaction and substrate disorder are also present. When applying to the fractional quantized Hall effect, we draw the conclusion that there must be a symmetry breaking in the many-body ground state. The possibility of writing the fractionally quantized Hall conductance as a topological invariant is also discussed.

Consequently we can equate σ with its average over all the phases $(0 \le \theta < 2\pi,\ 0 < \varphi \le 2\pi)$ that specify different boundary conditions, i.e.,

$$\sigma = \bar{\sigma} = \frac{e^2}{h} \int_0^{2\pi} \int_0^{2\pi} d\theta\, d\varphi \frac{1}{2\pi i} \left[\left\langle \frac{\partial \phi_0}{\partial \varphi} \middle| \frac{\partial \phi_0}{\partial \theta} \right\rangle - \left\langle \frac{\partial \phi_0}{\partial \theta} \middle| \frac{\partial \phi_0}{\partial \varphi} \right\rangle \right] . \tag{2.9}$$

$$\sigma = \bar{\sigma} = \frac{e^2}{hd} \sum_{K=1}^{d} \int_0^{2\pi} \int_0^{2\pi} d\theta\, d\varphi \frac{1}{2\pi i} \left[\left\langle \frac{\partial \phi_K}{\partial \varphi} \middle| \frac{\partial \phi_K}{\partial \theta} \right\rangle - \left\langle \frac{\partial \phi_K}{\partial \theta} \middle| \frac{\partial \phi_K}{\partial \varphi} \right\rangle \right] , \tag{3.1}$$

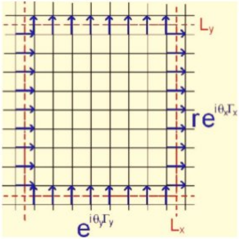
where d is the degree of the degeneracy, and $\{\phi_K\}$ is an

Gapless edge states for interacting systems

PHYSICAL REVIEW B **74**, 045125 (2006)

General theorem relating the bulk topological number to edge states in two-dimensional insulators

Xiao-Liang Qi,[1,2] Yong-Shi Wu,[3] and Shou-Cheng Zhang[2,1]
[1]*Center for Advanced Study, Tsinghua University, Beijing, 100084, China*
[2]*Department of Physics, McCullough Building, Stanford University, Stanford, California 94305-4045, USA*
[3]*Department of Physics, University of Utah, Salt Lake City, Utah 84112-0830, USA*
(Received 7 April 2006; published 26 July 2006)



$$H = H_{\Gamma_x\Gamma_y}(r,\theta_x,\theta_y). \quad (3)$$

$$A_\mu(r,\theta_x,\theta_y) = -i\langle G(r,\theta_x,\theta_y)|\frac{\partial}{\partial\theta_\mu}|G(r,\theta_x,\theta_y)\rangle, \quad (4)$$

$$N = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} d\theta_x d\theta_y F_{xy}(1,\theta_x,\theta_y)$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} d\theta_x d\theta_y \left(\frac{\partial A_y}{\partial\theta_x} - \frac{\partial A_x}{\partial\theta_y}\right)\Bigg|_{r=1}. \quad (6)$$

If N>0, then gapless edge states at r<1

Topological order

PHYSICAL REVIEW B VOLUME 41, NUMBER 13 1 MAY 1990

Ground-state degeneracy of the fractional quantum Hall states in the presence of a random potential and on high-genus Riemann surfaces

X. G. Wen*
Institute for Theoretical Physics, University of California-Santa Barbara, Santa Barbara, California 93106

Q. Niu
Department of Physics, University of California-Santa Barbara, Santa Barbara, California 93106

(Received 17 October 1989)

The fractional quantum Hall (FQH) states are shown to have $\bar{q}^{g}$-fold ground-state degeneracy on a Riemann surface of genus g, where $\bar{q}$ is the ground-state degeneracy in a torus topology. The ground-state degeneracies are directly related to the statistics of the quasiparticles given by $\theta=\bar{p}\pi/\bar{q}$. The ground-state degeneracy is shown to be invariant against weak but otherwise arbitrary perturbations. Therefore the ground-state degeneracy provides a new quantum number, in addition to the Hall conductance, characterizing different phases of the FQH systems. The phases with different ground-state degeneracies are considered to have different topological orders. For a finite system of size L, the ground-state degeneracy is lifted. The energy splitting is shown to be at most of order $e^{-L/\xi}$. We also show that the Ginzburg-Landau theory of the FQH states (in the low-energy limit) is a dual theory of the U(1) Chern-Simons topological theory.

The FQH states and chiral spin states are very special in the sense that their ground-state properties are not characterized by the symmetries in their ground states. The transition from one FQH state (or chiral spin state) to another is not associated with a charge in the symmetries of the states. In this paper we will demonstrate that the FQH states and chiral spin states contain nontrivial topological structures. The different FQH states and chiral spin states may be classified by topological orders.

Topological order

- Systems with topological order are characterised by quasi-particles with fractional quantum numbers and statistics.
- Their universal properties (quasi-particle quantum numbers and statistics) are described by topological field theories.
- Quantum number frationisation occurs in strongly correlated fermion systems (Hubbard models): Spinons and Holons of RVB theory. Can be understood in the framework of emergent gauge fields.

  G. Baskaran and P.W. Anderson, Phys. Rev. B 37, 580 (1988)

  T. Senthil and Fisher, Phys. Rev. B 62, 7850 (2000)

Outline

The Quantum Hall Effect and Topological order

Topological Insulators

Kitaev-Hubbard Model

Topological Insulators: Theory

Prediction of Insulating materials with metallic surfaces

PRL **95**, 146802 (2005) PHYSICAL REVIEW LETTERS week ending 30 SEPTEMBER 2005

Z_2 Topological Order and the Quantum Spin Hall Effect

C. L. Kane and E. J. Mele

RAPID COMMUNICATIONS

PHYSICAL REVIEW B **75**, 121306(R) (2007)

Topological invariants of time-reversal-invariant band structures

J. E. Moore[1,2] and L. Balents[3]

Three dimensional topological invariants for time reversal invariant Hamiltonians and the three dimensional quantum spin Hall effect

Rahul Roy

arXiv:cond-mat/0607531v3 [cond-mat.mes-hall] 21 Jul 2006

PRL **98**, 106803 (2007) PHYSICAL REVIEW LETTERS week ending 9 MARCH 2007

Topological Insulators in Three Dimensions

Liang Fu, C. L. Kane, and E. J. Mele

Topological Insulators: Real materials

ARTICLES
PUBLISHED ONLINE: 10 MAY 2009 | DOI: 10.1038/NPHYS1270
nature physics

Topological insulators in Bi_2Se_3, Bi_2Te_3 and Sb_2Te_3 with a single Dirac cone on the surface

Haijun Zhang[1], Chao-Xing Liu[2], Xiao-Liang Qi[3], Xi Dai[1], Zhong Fang[1] and Shou-Cheng Zhang[3]*

Topological insulators are new states of quantum matter in which surface states residing in the bulk insulating gap of such systems are protected by time-reversal symmetry. The study of such states was originally inspired by the robustness to scattering of conducting edge states in quantum Hall systems. Recently, such analogies have resulted in the discovery of topologically protected states in two-dimensional and three-dimensional band insulators with large spin–orbit coupling. So far, the only known three-dimensional topological insulator is Bi_xSb_{1-x}, which is an alloy with complex surface states. Here, we present the results of first-principles electronic structure calculations of the layered, stoichiometric crystals Sb_2Te_3, Sb_2Se_3, Bi_2Te_3 and Bi_2Se_3. Our calculations predict that Sb_2Te_3, Bi_2Te_3 and Bi_2Se_3 are topological insulators, whereas Sb_2Se_3 is not. These topological insulators have robust and simple surface states consisting of a single Dirac cone at the Γ point. In addition, we predict that Bi_2Se_3 has a topologically non-trivial energy gap of 0.3 eV, which is larger than the energy scale of room temperature. We further present a simple and unified continuum model that captures the salient topological features of this class of materials.

LETTERS
PUBLISHED ONLINE: 10 MAY 2009 | DOI: 10.1038/NPHYS1274
nature physics

Observation of a large-gap topological-insulator class with a single Dirac cone on the surface

Y. Xia[1,2], D. Qian[1,3], D. Hsieh[1,2], L. Wray[1], A. Pal[1], H. Lin[4], A. Bansil[4], D. Grauer[5], Y. S. Hor[5], R. J. Cava[5] and M. Z. Hasan[1,2,6]*

Recent experiments and theories have suggested that strong spin–orbit coupling effects in certain band insulators can give rise to a new phase of quantum matter, the so-called topological insulator, which can show macroscopic quantum-entanglement effects[1-7]. Such systems feature two-dimensional surface states whose electrodynamic properties are described not by the conventional Maxwell equations but rather by an attached axion field, originally proposed to describe interacting quarks[8-15]. It has been proposed that a topological insulator[2] with a single Dirac cone interfaced with a superconductor can form the most elementary unit for performing fault-tolerant quantum computation[14]. Here we present an angle-resolved photoemission spectroscopy study that reveals the first observation of such a topological state of matter featuring a single surface Dirac cone realized in the naturally occurring Bi_2Se_3 class of materials. Our results, work as a matrix material to observe a variety of topological quantum phenomena.

The topological-insulator character of BiSb[5,6] led us to investigate the alternative Bi-based compounds Bi_2X_3 (X = Se, Te). The undoped Bi_2Se_3 is a semiconductor that belongs to the class of thermoelectric materials Bi_2X_3 with a rhombohedral crystal structure (space group D^5_{3d} ($R\bar{3}m$); refs 17, 18). The unit cell contains five atoms, with quintuple layers ordered in the Se(1)–Bi–Se(2)–Bi–Se(1) sequence. Electrical measurements report that, although the bulk of the material is a moderately large-gap semiconductor, its charge transport properties can vary significantly depending on the sample preparation conditions[19], with a strong tendency to be n-type[20,21] owing to atomic vacancies or excess selenium. An intrinsic bandgap of approximately 0.35 eV is typically measured in experiments[22,23], whereas theoretical calculations estimate the gap to be in the range of 0.24–0.3 eV (refs 20, 24).

Time reversal in Quantum mechanics

$$\psi_2 = e^{-iH(t_2-t_1)}\psi_1$$

If there is an operator $\mathcal{T}$ such that,

$$\mathcal{T}\psi_1 = e^{-iH(t_2-t_1)}\mathcal{T}\psi_2$$

Then the system is time reversal invariant.

Time reversal for Bloch hamiltonians,

$$\mathcal{T}u(k)_\alpha = (\sigma^y\ u^*(k))_\alpha$$

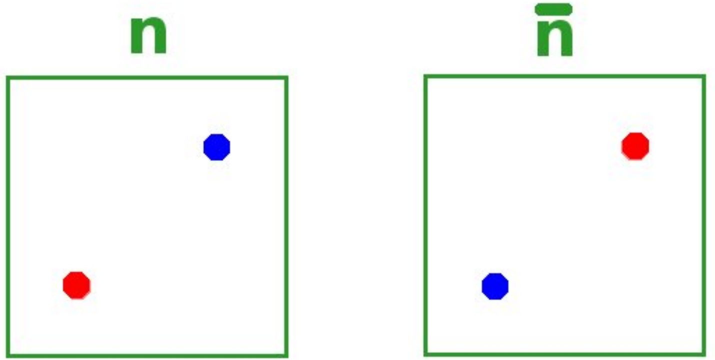
if

$$\sigma^y h^*(k)\sigma^y = h(-k)$$

Then system is time reversal invariant

Band Pairs

$$
\begin{aligned}
\mathcal{T} u_n(k) &= u_{\bar{n}}(-k) \\
\mathcal{T} u_{\bar{n}}(k) &= -u_n(-k) \\
\epsilon_n(k) &= \epsilon_{\bar{n}}(-k) \\
F^n_{ij}(k) &= -F^{\bar{n}}_{ij}(-k)
\end{aligned}
$$



The 2-d Z_2 invariant

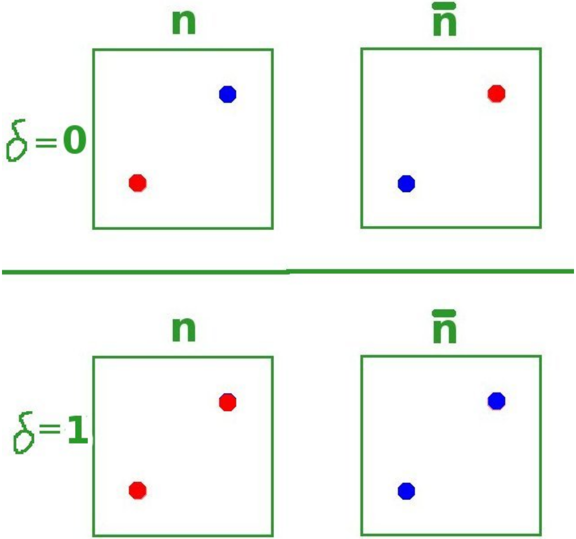
Consider a system with 2 occupied time reversed bands:

- The total Berry flux carried by them will always be zero.
- If the hamiltonian is smoothly perturbed, maintaining time reversal invariance, and they touch ($h(k)$ becomes degenerate), it will always happen at pairs of points $(k, -k)$.
- Time reversal invariance ensures that the flux exchanged by bands at these two points is always equal.
- The change of flux in each band is always even and hence the Chern index of each band modulo 2 is invariant under smooth time reversal symmetric perturbations.
- in general,

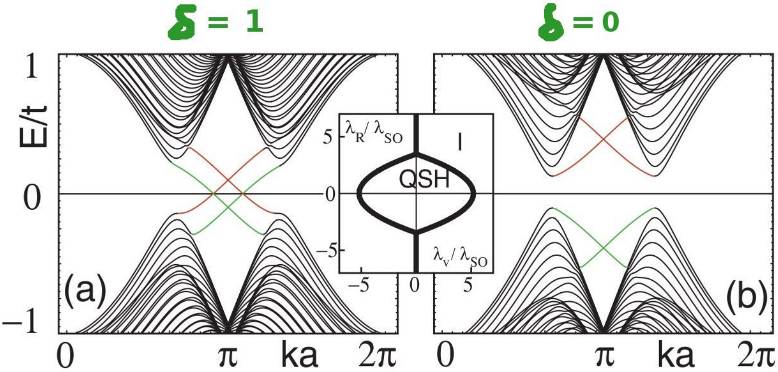
$$\delta = \left(\frac{1}{2} \sum_{n=1}^{N_B} |\nu_n| \right) \text{modulo } 2$$

is invariant.

The 2-d Z_2 invariant



Consequences at the edge



C. L. Kane and E. J. Mele PRL **95**, 146802 (2005)

- Even number of edge pairs (per edge) for $\delta = 0$
  Odd number of edge pairs (per edge) for $\delta = 1$
- No backscattering for a single pair due to time reversal symmetry.

The 3-d Z_2 invariants

Parameterise the 3-d torus by $-\pi \leq k_i \leq \pi, \; i = x, y, z$.

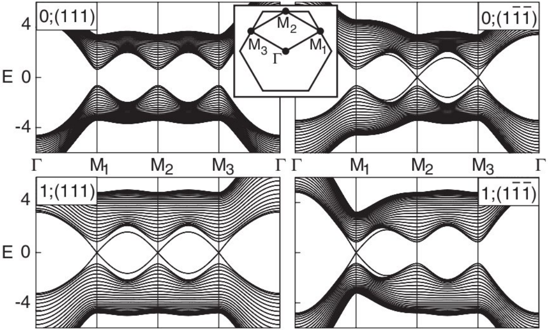
- There are many time reversal invariant planes. eg $k_i = 0, \pi$.
- The 2-d Z_2 invariants of these planes are all topological invariants.
- Of these, there are 4 independent invariants which can be chosen to be,

$$\delta_x \equiv \delta_{k_x=\pi}, \;\; \delta_y \equiv \delta_{k_y=\pi}, \;\; \delta_z \equiv \delta_{k_z=\pi}$$

$$\delta_0 \equiv \delta_{k_z=0}\delta_{k_z=\pi}$$

- If $\delta_0 = -1$, then “strong topological insulator".

Consequences at the surface



Liang Fu, C. L. Kane, and E. J. Mele PRL **98,** 106803 (2007)

- Even number of surface Dirac cones for $\delta = 0$
  Odd number of surface Dirac cones for $\delta = 1$
- Gaplessness for odd number of Dirac points protected by time reversal symmetry.

Outline

The Quantum Hall Effect and Topological order

Topological Insulators

Kitaev-Hubbard Model

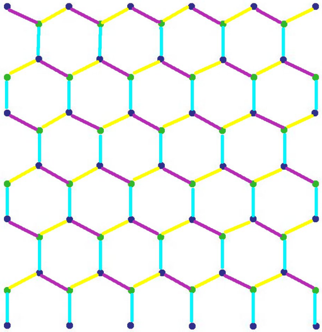
Collaborators

Syed R. Hassan
Sandeep Goyal
Archana Mishra
P.V. Sriluckshmy
(IMSc., Chennai)

David Senechal
(University of Sherbrooke)

The Kitaev Honeycomb Model

Alexei Kitaev,“Anyons in an exactly solved model and beyond", Ann. Phys. (N.Y) **321**, 2 (2006).



$$H = J_x \sum_{<ij>} \sigma_i^x \sigma_j^x + J_y \sum_{<ij>} \sigma_i^y \sigma_j^y + J_z \sum_{<ij>} \sigma_i^z \sigma_j^z$$

The Kitaev Honeycomb Model

- Simple exact solution. Same degree of difficulty as 1-d transverse field Ising model.
- Solution is a spin liquid with topological order, spinons and non-abelian anyons.

Open problem:
What is the ground state wave function in the spin basis ?

$$|GS\rangle = \sum_{x_1,\ldots,x_N} \Psi(x_1, x_2, \ldots, x_N)|x_1, x_2, \ldots, x_N\rangle$$

A proposal for engineering the model

VOLUME 91, NUMBER 9 PHYSICAL REVIEW LETTERS week ending 29 AUGUST 2003

Controlling Spin Exchange Interactions of Ultracold Atoms in Optical Lattices

L.-M. Duan,[1] E. Demler,[2] and M. D. Lukin[2]

[1]*Institute for Quantum Information, California Institute of Technology, mc 107-81, Pasadena, California 91125, USA*

[2]*Physics Department, Harvard University, Cambridge, Massachusetts 02138, USA*

(Received 25 October 2002; published 26 August 2003)

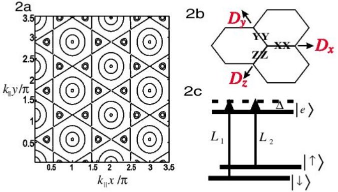


FIG. 2 (color online). (a) The contours with the three potentials in the form of Eq. (5). The minima are at the centers of the triangles when $\varphi_0 = \pi/2$. (b) The illustration of the model Hamiltonian (4). (c) The schematic atomic level structure and the laser configuration to induce spin-dependent tunneling.

Hamiltonian is then given by

$$H = -\sum_{\langle ij\rangle\sigma}(t_{\mu\sigma}a_{i\sigma}^\dagger a_{j\sigma} + \text{H.c.}) + \frac{1}{2}\sum_{i,\sigma}U_\sigma n_{i\sigma}(n_{i\sigma} - 1) + U_{\uparrow\downarrow}\sum_i n_{i\uparrow}n_{i\downarrow}, \quad (1)$$

Here $\langle i, j\rangle$ denotes the near neighbor sites in the direction μ, $a_{i\sigma}$ are bosonic (or fermionic) annihilation operators, respectively, for bosonic (or fermionic) atoms of spin σ localized on-site i, and $n_{i\sigma} = a_{i_\sigma}^\dagger a_{i_\sigma}$.

The Kitaev-Hubbard Model

$$H = \sum_{\langle ij \rangle_a} \left(C_i^\dagger \left(\frac{t + t' \sigma^a}{2} \right) C_j + h.c \right) + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

$U \to \infty$, Half filling:

$$H_{eff} = \sum_{\langle ij \rangle_a} \left(\frac{t^2 - t'^2}{U} \vec{S}_i \cdot \vec{S}_j - \frac{2t'^2}{U} S_i^a S_j^a \right)$$

(similar to) J. Chaloupka, George Jackeli and Giniyat Khaliullin, Phys. Rev. Lett. 105, 027204 (2010)

- t'=0, Heisenberg Model
- t'=t, Kitaev Model

The spin dependent hopping term

The t' term breaks time reversal symmetry.

$$C_i^\dagger \left(\frac{t + t'\sigma^a}{2} \right) C_j + h.c \rightarrow T \rightarrow C_i^\dagger \left(\frac{t - t'\sigma^a}{2} \right) C_j + h.c$$

A time reversal symmetric term,

$$C_i^\dagger \left(\frac{t + it'\sigma^a}{2} \right) C_j + h.c \rightarrow T \rightarrow C_i^\dagger \left(\frac{t + it'\sigma^a}{2} \right) C_j + h.c$$

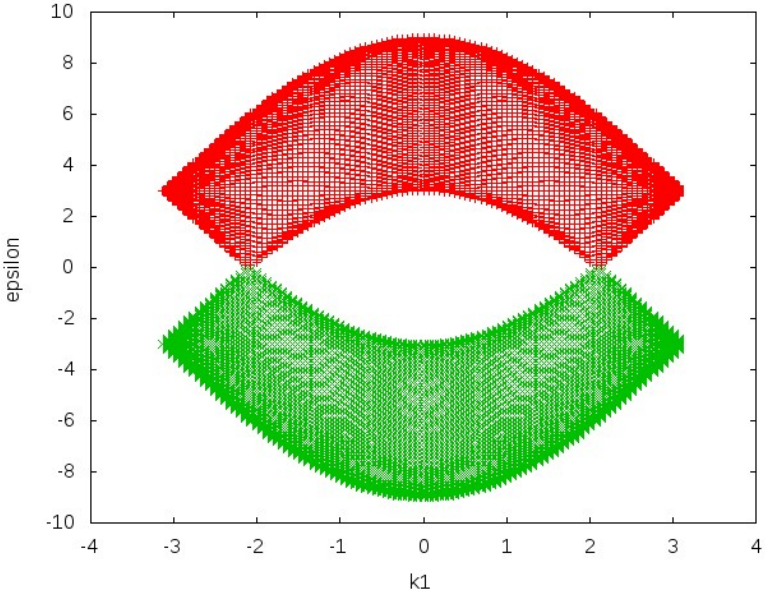
Does not give the Kitaev model at large U, half filling.

$U = 0$

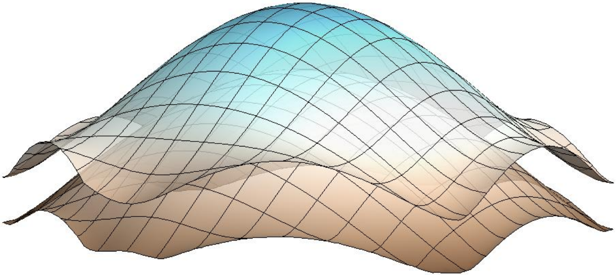
$$
\begin{aligned}
h(k) &= \begin{pmatrix} 0 & \Sigma(k) \\ \Sigma^\dagger(k) & 0 \end{pmatrix} \\
\Sigma(k) &= \frac{t}{2} f(k) + \frac{t'}{2}\left(\sigma^z + e^{-ik_1}\sigma^x + e^{ik_2}\sigma^y\right) \\
f(k) &= 1 + e^{-ik_1} + e^{ik_2}
\end{aligned}
$$

$$
\begin{aligned}
\epsilon(k) &= \pm\sqrt{\epsilon_0^2 + \frac{3t'^2}{4} + \pm\frac{t'}{2}|\vec{B}(k)|} \\
\vec{B} &= \left(1 - t' \sin k_1 + \cos k_2 + \cos k_3\right)\hat{x} \\
&\quad + \left(1 + \cos k_1 - t' \sin k_2 + \cos k_3\right)\hat{y} \\
&\quad + \left(1 + \cos k_1 + \cos k_2 - t' \sin k_3\right)\hat{z}
\end{aligned}
$$

$U = 0, \ t' = 0.0$

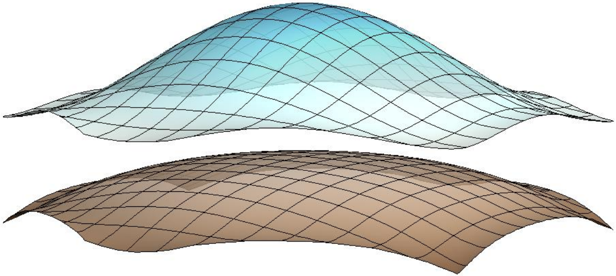


$U = 0,\ t' = 0.1$



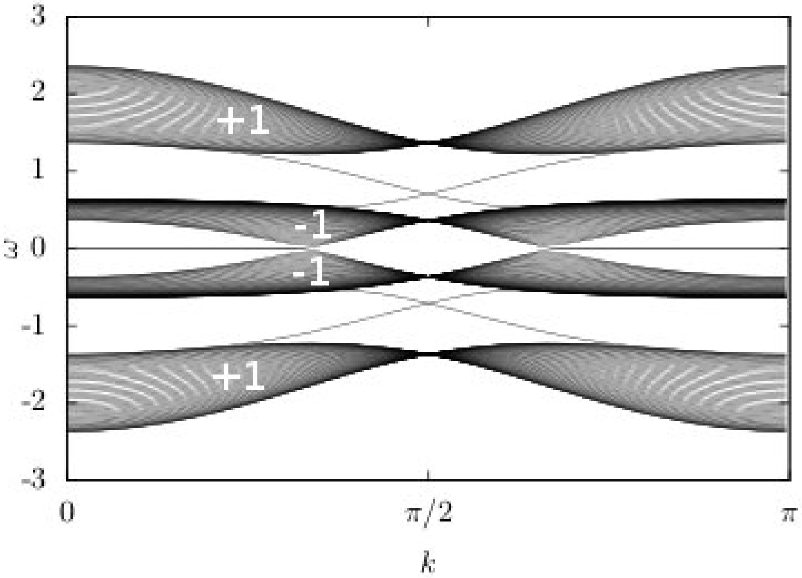
The bands overlap for $t' < \sqrt{7} - \sqrt{6} = 0.717$

$U = 0,\ t' = 0.9$

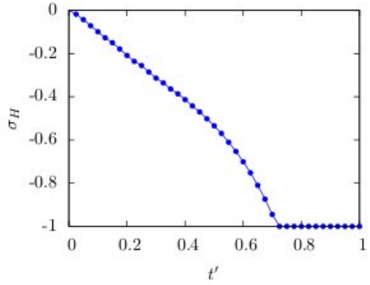


The bands overlap for $t' < \sqrt{7} - \sqrt{6} = 0.717$

$U = 0$: Chern Numbers



“Hall conductance" at $1/4$ filling



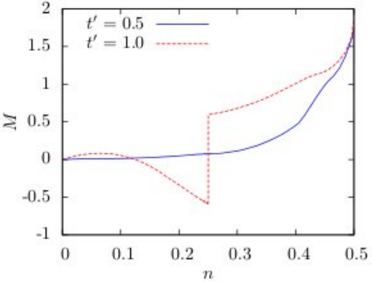
The Pancharatnam-Berry curvature for each band is given by

$$PB^{pp'}(k) = \frac{\epsilon_{ij}}{8\pi i}\left(\partial_i \Phi^{pp'}(k)^\dagger \partial_j \Phi^{pp'}(k) - \mathrm{H.c.}\right)$$

The “Hall conductivity" is

$$\sigma_H = \int \frac{d^2k}{(2\pi)^2} \sum_{pp'} \Theta(\mu - \epsilon_{pp'}(k)) PB^{pp'}$$

Angular momentum



The Orbital magnetization of Bloch electrons of the band pp' is

$$\begin{aligned} M_{pp'} &= \frac{e}{2\hbar}\int \frac{d^2k}{(2\pi)^2}\Theta(\mu - \epsilon_{pp'}(k)) \\ &\times \langle \partial_{\mathbf{k}}\Phi^{pp'}|(H_{\mathbf{k}} + \epsilon_{pp'}(\mathbf{k}) - 2\mu)|\partial_{\mathbf{k}}\Phi^{pp'}\rangle, \end{aligned}$$

Interactions

Is the physics stable against interactions ?

- Does the gap persist ?
- Do the chiral edge states persist ?

Variational Cluster Approximation

Cluster Perturbation Theory:

$$H = H_c + T$$

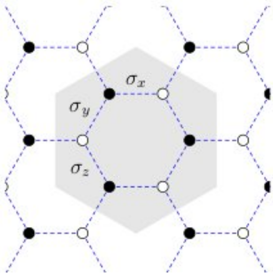
$$G(\tilde{k}, \omega) \approx \left(G_c^{-1}(\omega) + T(\tilde{k}) \right)^{-1}$$

Variational Cluster Approximation:

$$\Omega(h) \approx \Omega_c(h) + \int \frac{d\omega}{\pi} \frac{d^2\tilde{k}}{(2\pi)^2} \ln \det \left(1 - T(\tilde{k}) G_c(\tilde{k}, i\omega) \right)$$

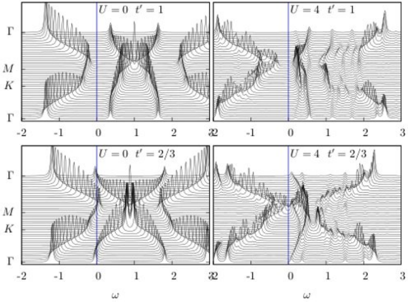
Minimize the grand potential with respect to the variational parameters h.

The cluster

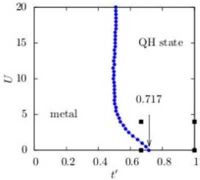


Variational parameters: M the total magnetization and μ_c, the cluster chemical potential.

The gap

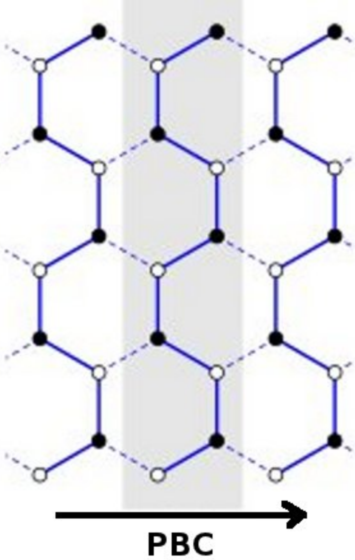


$U=0$ $t'=1$
$U=4$ $t'=1$
$U=0$ $t'=2/3$
$U=4$ $t'=2/3$
Γ
M
K
Γ
-2 -1 0 1 2 3
ω

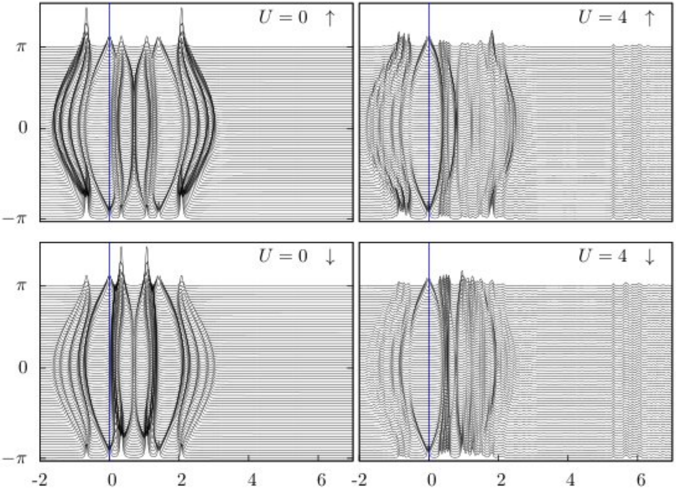


QH state
metal
0.717
U
t'
0 5 10 15 20
0 0.2 0.4 0.6 0.8 1

The edge states



The edge states



AFQHE: Warm up problem

- Haldane model with nearest neigbour interactions,

$$H_{int} = V \sum_{ij} n_i n_j$$

- Mean field theory:

$$n_i n_j \approx \left(\chi_{ij} C_i^\dagger C_j + h.c \right) - |\chi_{ij}|^2$$

- We find solutions corresponding to $\nu = 1/3$ (one third of lower band occupied).
- Interpretation in terms of composite fermions: Wave function ?
- Generalising to Kitaev-Hubbard using VCPT ?

$t' = 0$ at half filling

Vol 464 | 8 April 2010 | doi:10.1038/nature08942 nature

ARTICLES

Quantum spin liquid emerging in two-dimensional correlated Dirac fermions

Z. Y. Meng[1], T. C. Lang[2], S. Wessel[1], F. F. Assaad[2] & A. Muramatsu[1]

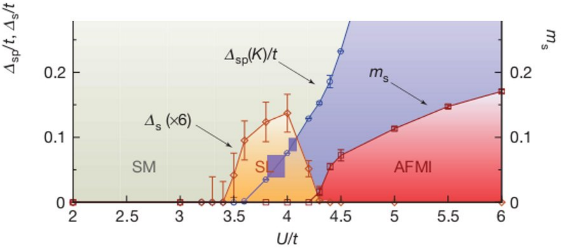


Figure 1 | Phase diagram for the Hubbard model on the honeycomb lattice at half-filling. The semimetal (SM) and the antiferromagnetic Mott insulator (AFMI) are separated by a gapped spin-liquid (SL) phase in an intermediate-coupling regime. $\Delta_{sp}(K)$ denotes the single-particle gap and Δ_s denotes the spin gap; m_s denotes the staggered magnetization, whose saturation value is 1/2. Error bars, s.e.m. Inset, the honeycomb lattice with

$t' = 1$

At $U = 0$, Dirac points persist at all t'
At $U/t >> 1$,

$$H = H_K - \lambda \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

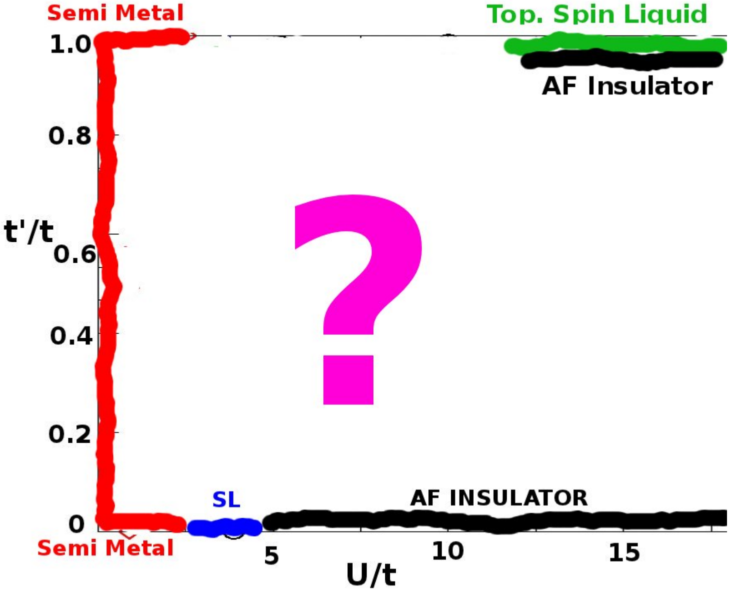
J. Chaloupka, George Jackeli and Giniyat Khaliullin, Phys. Rev. Lett. 105, 027204 (2010)

$$H = H_K - \lambda \sum_{\langle ij \rangle} S_i^z \cdot S_j^z$$

S. Mandal, Subhro Bhattacharjee, K. Sengupta, R. Shankar and G. Baskaran, Physical Review **B 84**, 155121 (2011)

- The Spin Liquid remains till $\lambda \approx 0.1$
- From $\lambda \approx 0.1$ to $\lambda \approx 1$, an Antiferromagnetic phase $q = (0, \pi)$.
- $\lambda > 1$, Neel phase $q = (0, 0)$.

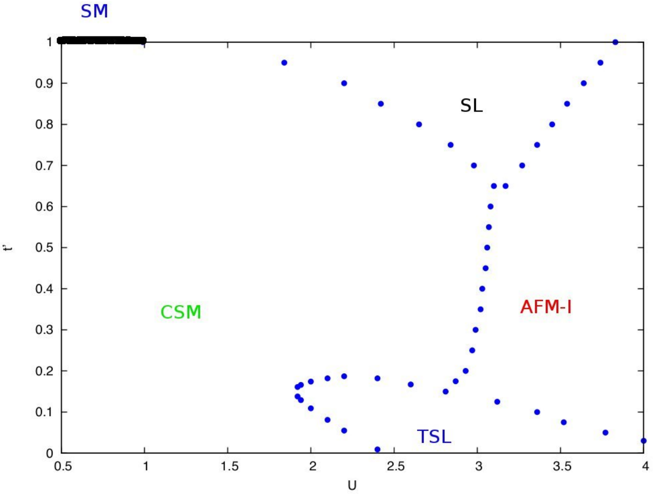
The question



Time reversal symmetry

- Kitaev model ($U \to \infty$) is time reversal symmetric (TRS) but microscopic theory is not. What happens at intermediate U ?
- At $t' = 0$, system is time reversal symmetric (TRS). Can the spin liquid near $t' = 0$ be continously connected to the spin liquid near $t' = 1$?
- Result: Particle-Hole symmetry in this model implies that the Mott phase is TRS. Proof to all orders in $t/U, \; t'/U$ perturbation theory

The Phase Diagram



Future work

- Nature of the spin liquid phases at half-filling.
- AFQHE states at less than quarter filling.