Aspects of higher-dimensional partitions

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Talk at IMSc on March 14, 2012

Based on: 1. Srivatsan Balakrishnan, SG and Naveen S. Prabhakar arXiv:1105.6231;
2. SG (to appear);
3. N. Destainville and S.G. (to appear).

Thanks to the High Performance Computing Environment at IITM

Partitions of a positive integer

- How many ways can a positive integer can be written as sum of non-zero positive integers? Call it $p_1(n)$.

$$4 \quad 3\,1 \quad 2\,2 \quad 2\,1\,1 \quad 1\,1\,1\,1 \implies \boxed{p_1(4) = 5}\,.$$

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- A formal definition: A partition of an integer, n, is a weakly decreasing sequence of non-zero integers $(a_1, a_2, \ldots)$ such that
  - $\sum_i a_i = n$ and $a_{i+1} \leq a_i$ for all i.
- [Euler] Let $P_1(q) := 1 + \sum_{n=1} p_1(n)\, q^n$. Then

$$P_1(q) = \prod_{m=1}^{\infty} (1 - q^m)^{-1} \,.$$

- What is $p_1(200)$?

Partitions of a positive integer

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$$P_1(q) = \prod_{m=1}^{\infty} (1-q^m)^{-1}\,.$$

- [MacMahon] $p_1(200) = 397\ 29990\ 29388$.

  [Hardy-Ramanujan] $p_1(n) \sim \exp(\pi\sqrt{2n/3})$ or $\log p(n) \sim \sqrt{n}$.

Higher-dimensional partitions

- A d-*dimensional partition* of n is defined to be a map from $\mathbb{Z}^d_{>0}$ to $\mathbb{Z}_{\geq 0}$ such that it is weakly decreasing along all directions and the sum of all its entries add to n.
- Let us denote the partition by the hypermatrix $(a_{i_1,i_2,\ldots,i_d})$.
- The weakly decreasing condition along the r-th direction implies that

$$a_{i_1,i_2,\ldots,i_r+1,\ldots,i_d} \leq a_{i_1,i_2,\ldots,i_r,\ldots i_d} \quad \forall\, (i_1, i_2, \ldots, i_d)\ .$$

- Let us denote the d-dimensional partition of n by $p_d(n)$. Thus $p_1(n)$ will refer to the usual partition of n.
- Two-dim. partitions are also called plane partitions.
- Three-dim. partitions are also called solid partitions.

Plane partitions

- Plane partitions can thus be written out as a two-dimensional array of numbers, (a_{ij}).
- For instance, the two-dimensional partitions of 4 are

$$
4 \qquad 3\ 1 \qquad \begin{matrix} 3 \\ 1 \end{matrix} \qquad 2\ 2 \qquad \begin{matrix} 2 \\ 2 \end{matrix} \qquad 2\ 1\ 1 \qquad \begin{matrix} 2 & 1 \\ 1 & \end{matrix} \qquad \begin{matrix} 2 \\ 1 \\ 1 \end{matrix}
$$

$$
1\ 1\ 1\ 1 \qquad \begin{matrix} 1 & 1 & 1 \\ 1 & & \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 1 & \\ 1 & \end{matrix} \qquad \begin{matrix} 1 \\ 1 \\ 1 \\ 1 \end{matrix} \qquad \Longrightarrow \quad \boxed{p_2(4) = 13}
$$

- [MacMahon] Let $P_2(q) := 1 + \sum_{n=1} p_2(n)\, q^n$. Then

$$
P_2(q) = \prod_{m=1}^{\infty} (1 - q^m)^{-m} \ .
$$

- Took MacMahon about 20 years to prove his conjecture!
- $p_2(200) = 40\ 66263\ 49006\ 862301\ 69190\ 82185$.

Where do these objects appear

Higher-dimensional partitions appear in several different areas of physics, mathematics and computer science. I list a few

- The infinite state Potts model in $(d+1)$ dimensions gets related to d-dimensional partitions in the high temperature limit;
- in the study of directed compact lattice animals;
- in the counting of BPS states in string theory and supersymmetric field theory. For instance, it is known that the numbers of mesonic and baryonic gauge invariant operators in some $\mathcal{N}=1$ supersymmetric field theories get mapped to higher-dimensional partitions.
- The Gopakumar-Vafa (Donaldson-Thomas) invariants (in particular, the zero-brane contributions) are also related to deformed versions of higher-dimensional partitions (usually plane partitions).

Counting of BPS states

- The generating function of elect. charged $\frac{1}{2}$-BPS states in the het. string compactified on a six-torus is given by $(\frac{1}{2}\mathbf{q}_e^2 := n-1)$

$$\sum_{n=0}^{\infty} d(n) q^n = \frac{16}{q\prod_{n=1}^{\infty}(1-q^n)^{24}} = \frac{16}{\eta(\tau)^{24}} \,.$$

- The generating function of Donaldson-Thomas (or Gopakumar-Vafa) invariants on the non-commut. conifold is given by ($q \sim e^{-g_s}$; t – Kähler modulus) [Szendroi,Young]

$$\prod_{n=1}^{\infty}(1-q^n)^{-n}(1-e^{-t}q^n)^{-n}(1-e^{+t}q^n)^{-n}$$

Note the appearance of the Euler and MacMahon generating functions.

The problem when $d > 2$

- MacMahon proposed a generating function for d-dimensional partitions

$$M_d(q) = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)^{\binom{n+d-2}{d-1}}} := \sum_{n=0}^{\infty} m_d(n)\, q^n\ .$$

- Is $M_d(q) = P_d(q)$? In 1967, it was shown to fail for $d > 2$ i.e., the d-dim MacMahon number $m_d(6) \neq p_d(6)$.
- A gen. function for d-dim ($d > 2$) partitions is not known and it appears that a simple formula does not exist.
- Is brute force is the only way out? What is the smallest amount of computing resource that is needed to enumerate $p_{100}(25)$? This is the question that we address.

Plan of talk

- Introduction (just completed)
- Part I: Refined coating of higher-dimensional partitions.
- Part II: Asymptotics of higher-dimensional partitions.
- Concluding remarks

Part I

Refined counting of higher-dimensional partitions

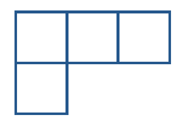
Credits: HPCE at IITM for providing me access to a world class supercomputing facility.

Suresh Govindarajan, *Notes on higher-dimensional partitions*, to appear soon on the arXiv.

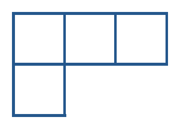
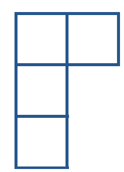
Ferrer's diagrams – partitions as pictures

Euler: The no. of partitions of n into at most with m parts is equal to the no. of partitions of n into parts $\leq m$.

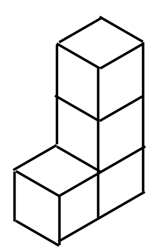
- $3\ 1$ is a partition of 4 with two parts. Ferrer's associated a diagram as follows:

$3\ 1 \quad \longleftrightarrow$ 

- Ferrer's bijection: (conjugation or xy exchange)

$(3\ 1) =$  $\longleftrightarrow$  $(= 2\ 1\ 1)$

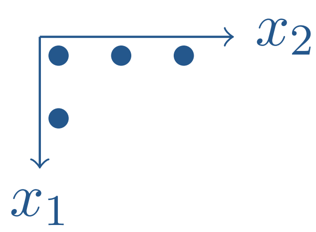
- The plane partition of 4 can be rep. as a pile of cubes:

$\begin{matrix} 3 \\ 1 \end{matrix} \quad \longleftrightarrow$ 

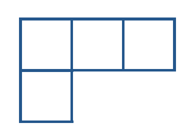
An alternate definition of partitions

- An unrestricted d-dimensional partition of n is a collection of n points (nodes) in $\mathbb{Z}_+^{d+1}$ satisfying the following property: if the collection contains a node $\mathbf{a} = (a_1, a_2, \ldots, a_{d+1})$, then all nodes $\mathbf{x} = (x_1, x_2, \ldots, x_{d+1})$ with $0 \leq x_i \leq a_i \ \forall$ $i = 1, \ldots, d+1$ also belong to the collection.
- We call the collection of nodes a $d+1$-dim Ferrers diagram.
- For instance, the one-dimensional partition of 4 can be written as

$$\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\} \text{ or } \left(\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{smallmatrix} \right) \text{ in compressed form },$$



or



Conjugation in higher-dimensional partitions

- The natural analog of conjugation for a $d+1$-dim. Ferrers Diagram (FD) is to permute the $(d+1)$ axes of the FD.
- We can organize Ferrers Diagrams by the studying the action of S_{d+1}.
- For instance, all plane partitions of 4 can be written as

$$\boxed{4 \qquad \begin{array}{c}1\\1\\1\\1\end{array} \qquad 1\ 1\ 1\ 1} \qquad \boxed{\begin{array}{cc}2 & 1\\ 1 & \end{array}} \qquad \boxed{\begin{array}{cc}1 & 1\\ 1 & 1\end{array} \qquad 2\ 2 \qquad \begin{array}{c}2\\2\end{array}}$$

$$\boxed{3\ 1 \qquad \begin{array}{c}3\\1\end{array} \qquad 2\ 1\ 1 \qquad \begin{array}{c}2\\1\\1\end{array} \qquad \begin{array}{ccc}1 & 1 & 1\\ 1 & & \end{array} \qquad \begin{array}{cc}1 & 1\\ 1 & \\ 1 & \end{array}}$$

- Up to S_3 action, there are really only four types of plane partitions of 4.
- The order of cosets of S_3/H for $H = S_3, S_2, S_1$ is $1, 3, 6$.

An important observation

- Treat partitions in all dimensions on the same footing keeping n fixed.
- Recall that the FD's for $p_d(n)$ will be obtained by a collection of of n nodes in $(d+1)$ dimensions.
- What happens when $d+1>n$? Consider $n=2$ – there are three plane partitions related to each other by the action of S_3 – however, the nodes always extend along only one axis.

$$\boxed{\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}$$

- Thus, there is really one FD, if we take into account the action of S_3 (more generally, S_{d+1}). The symmetry of the GFD is S_2 (S_d). Thus, we see that $p_d(2)$ is given by $\text{order}(S_{d+1}/S_d) = d+1$.

The intrinsic dimension of an FD

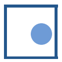
- Given an FD, let it be contained in a r-dimensional hyperplane but not in any $(r-1)$-dimensional hyperplane. The intrinsic dimension(id) of the FD is defined to be r.
- In the previous example, the id $r = 1$.
- The subgroup S_{d+1-r} that permutes the other coordinates leaves the FD unchanged.
- A FD of id r may have a further symmetry $H \subseteq S_r$ that acts trivially on it.
- Thus the number of distinct FD's obtained by the action of S_{d+1} on a FD is in one to one correspondence with the elements of the coset $S_{d+1}/(S_{d+1-r} \times H)$. The number of coset elements is

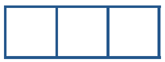
$$\frac{(d+1)!}{(d+1-r)! \times \text{dim}(H)} = \binom{d+1}{r} \times \frac{r!}{\text{ord}(H)} := \binom{d+1}{r} \times \text{weight} \ .$$

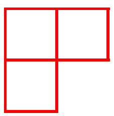
Generalized FD's

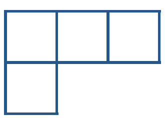
- Let us say that an FD is strict if its dimension equals its intrinsic dimension.
- Let us denote by gFD the equivalence class of strict FD's of i.d. r under the action of S_r.
- Given a strict FD λ, the number of FD's in its equivalence class is then given by its weight, i.e., $\frac{r!}{\text{ord}(H)}$, where H is the symmetry of λ.
- The first refinement that one can consider is to count the number of strict FD's.
- This was first observed and considered by Atkin et. al. in 1967 but virtually nothing has happened since.

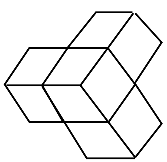
Examples


The gFD is taken to have $r = 0$, $n = 1$ and $w = 1$.


The gFD has $r = 1$, $n = 3$ and $w = 1$. More generally, we can see that all GFD's with $r = 1$ arise as k boxes (for some $k > 1$) in a line.


The gFD has $r = 2$, $n = 3$ and $w = 1$. It is easy to see that there are only two GFD's with size $n = 3$.


The gFD has $r = 2$, $n = 4$ and $w = 2$.


This gFD has $r = 3$, $n = 4$ and $w = 1$.

Remark: Thus, the problem of enumerating higher-dimensional partitions of n is reduced to enumerating all strict FD's with n nodes and id $r < n$.

The binomial transform

- Let a_{nr} denote the number of strict FD's with n-nodes and id r. Let $A = (a_{nr})$ be the corresponding matrix. Then

$$a_{nr} := \sum_{\text{gFDs with id } r\ \& n \text{ nodes}} w(\text{gFD})\ .$$

- Note that $a_{nr} = 0$ when $r \geq n$. It is a lower-triangular matrix.
- This leads to an interesting formula $p_d(n)$. [Atkin et. al. 1967]

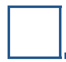
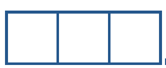
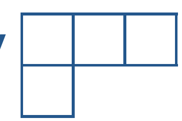
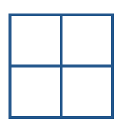
$$p_d(n) = \sum_{r=0}^{n-1} \binom{d+1}{r} a_{nr}\ .$$

- An example:

$$p_d(3) = \binom{d+1}{1} w\left(\square\square\square\right) + \binom{d+1}{2} w\left(\begin{array}{cc}\square & \square \\ \square & \end{array}\right)$$

$$= \binom{d+1}{1} + \binom{d+1}{2}\ .$$

Properties of the matrix A

- $a_{n0} = \delta_{n,1}$ – this follows since there is precisely one gFD with $id = 0$: . It has $n = 1$.
- $a_{r+1,r} = 1$ for all $n \geq 1$ – again there is only one gFD of size $(r+1)$ and id r.
- $a_{n1} = 1$ for $n > 1$ – this follows there is only one gFD of size $n > 1$ and id 1 – this is the one with n boxes in a single row like .
- $a_{n,n-2} = (n-1)(n-2)/2$. There are two gFD's with size n and id $(n-2)$. When $n = 4$, they are given by  which has weight 2 and  which has weight 1. For $n > 4$, the corresponding gFD's have weight $\binom{n-2}{1}$ and $\binom{n-2}{2}$ respectively.

A second combinatorial interpretation for the A-matrix

- it is easy to see that $a_{r+1,r} = 1$ – one needs to add r nodes to the first one at the origin such that all r-dimensions of the FD are used to get strict FD. These have to be nodes of the form $(1, 0, \ldots, 0)^T$ up to S_r-action – there are precisely r of them. Call this unique strict FD of id r, μ_r.

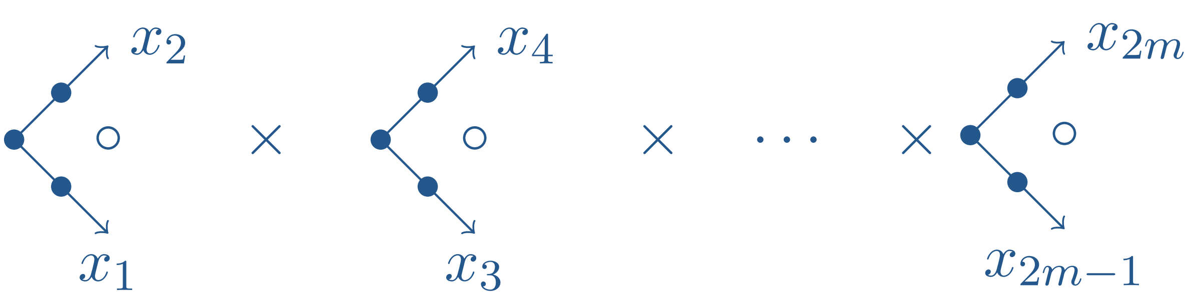
$$\mu_r := \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & & 1 \end{pmatrix} .$$

- All strict FD's of id r must contain all $r+1$ nodes in μ_r. If λ is a FD of id r, then the skew $FD\ \lambda \setminus \mu_r$ contains $m = n - r - 1$ nodes. Equivalently, the FD λ can be obtained by adding m-nodes to μ_r.

$a_{m+r+1,r}$ is the number of strict FD's obtained by adding m-nodes to μ_r.

The reduced dimension

- Consider the skew FD $\lambda \setminus \mu_r$ with m nodes. Let the m-nodes be contained in a x-dim hyperplane but not in any $x-1$-dim hyperplane. We call this the reduced dimension or rd of the skew FD as well as the FD.
- What is the maximum value of x for a skew FD with m nodes is $2m$. A pictorial proof.



- Thus, given an FD λ – we now several new attributes – its id, rd as well as the number of nodes in $\lambda \setminus \mu_r$. Can we further refine the counting?

The second transform

- We say that a skew FD $\lambda \setminus \mu_r$ is strict if it rd and id are equal.
- Let $c_{m,x}$ denote the number of strict skew FD's with m nodes and rd x. Then one has

$$a_{r+m+1,r} = \sum_{x=1}^{2m} \binom{r}{x} c_{m,x}$$

  The binomial factor $\binom{r}{x}$ takes care of the superfluous $(r-x)$ dimensions for an FD obtained from a strict skew FD by adding nodes from μ_r for $r > x$ to it.
- The transform implies that $g_m(r) := 2m!!\, a_{m+r+1,r}$ is a polynomial of degree $2m$ in r (conjecturally) with integral coefficients. When known, this polynomial determines all entries in the matrix A on the m-shifted diagonal.

A recap

- $$p_d(n) \longrightarrow a_{n,r} \longrightarrow c_{m,x}$$
- Let us now revisit the problem of determining $p_{100}(25)$. We only need to do is to determine all elements in one row of the A-matrix i.e., $a_{25,*}$. This needs us to generate 23 numbers.
- This can be done, for instance, by computing partitions of 25 for $d = 1, \ldots, 23$. There exist two algorithms, one due to Bratley and McKay(BM) and another due to Knuth that can be used. However, it is *impossible* to do so in a reasonable amount of time.
- It is clearly better to directly evaluate entries in the matrix A, if possible. We have shown that a simple modification of the BM algorithm can be used to directly enumerate entries in the A-matrix. This is not good enough.

Recap – continued

- One can then use the C-matrix to compute entries in shifted diagonals. We do not have an efficient algorithm to enumerate this. So we used the modified BM algorithm that generates all strict FD's that contribute to the A-matrix and binned them by measuring their rd.
- This is highly inefficient but enabled us to generate entries with shifts $m = 0, 1, \ldots, 8$. We however managed to enumerate all entries in the first 23 rows of A.
- This implies that we need to generate fifteen entries $a_{25,1}$ to $a_{25,15}$ that remains just out of reach.
- Clearly, we need further theoretical inputs and possibly computational inputs.

The matrices A and C

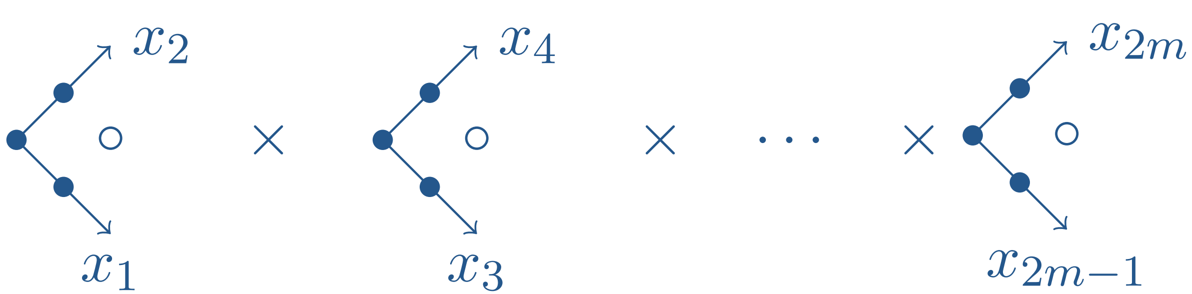
$$A = \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 0 & 1 & 3 & 1 & & \\ 0 & 1 & 5 & 6 & 1 & \\ 0 & 1 & 9 & 18 & 10 & 1 \end{pmatrix}.$$

$$C = \begin{pmatrix} 1 & 1 & & & & & & & & \\ 1 & 3 & 6 & 3 & & & & & & \\ 1 & 7 & 20 & 46 & 45 & 15 & & & & \\ 1 & 11 & 61 & 198 & 480 & 645 & 420 & 105 & & \\ 1 & 18 & 138 & 706 & 2508 & 6441 & 10395 & 9660 & 4725 & 945 \end{pmatrix}.$$

$$g_2(r) = r(r+1),\ g_3(r) = r(6 - r + 2r^2 + r^3)$$

Further refinements

- The idea to identify some simple and easily computable part of the computation of the A or C matrices that can be removed preferably by a transform.
- Due to some conjectures, we had indirect evidence that for fixed m, only half of the $2m$ entries in $c_{m,*}$ are needed.
- Suppose a skew FD, $\lambda \setminus \mu_r$, splits into two components – then we may not need to recount them as they would have appeared in another counting.
- In order to make this precise, we say that a skew FD is reducible it the set of notes can be broken into two disjoint sets such that each set lies in orthogonal hyperplanes.



$$\equiv (\sigma_2)^n$$

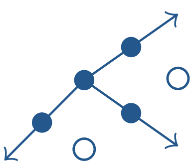
The third transform

- This transform removes all reducible components isomorphic to σ_2 and leads to a new triangle $D = (d_{m,x})$ (with $d_{0,0} \equiv 1$). There are $(2y-1)!!$ strict skew FD's isomorphic to σ_2^y in $2y$ dimensions.

$$c_{m,x} = \sum_{y=2x-3m}^{m} \binom{x}{2y} (2y-1)!!\, d_{m-y,x-2y} \ .$$

- For fixed m, it has $[3m/2]$ entries instead of the $2m$ entries for the corresponding row in C.
- Contributions to $d_{2m,3m}$ turn out to reducible of the form σ_3^m where

$$\sigma_3 \equiv$$



$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \ .$$

The final transform

- Clearly the natural step is to remove all reducible components isomorphic to σ_3 but it turns out that is not enough.
- Let $\mathcal{D} = \cup_r \mathcal{D}_r$, where $\mathcal{D}_r$ denotes the set of strict FD's of id r that consist of nodes of type $(1,0,\ldots,0)^T$ and $(1,1,0,\ldots,0)^T$ and its images under S_r.
- Let $e_{m,r}$ denote the number of elements of $\mathcal{D}_r$ obtained by adding m-nodes to μ_r. Then, $e_{m,r} = \binom{\binom{r}{2}}{m}$.
- The following transform removes all reducible components that lie in $\mathcal{D}$ and introduces our last matrix $F = (f_{n,r})$.

$$a_{m+r+1,r} = \sum_{x=1}^{r} \sum_{p=0}^{m} \binom{r}{x} \binom{\binom{r-x}{2}}{m-p} f_{p+x+1,\,x}\ .$$

The final transform

- Is the following transform new?

$$a_{m+r+1,r} = \sum_{x=1}^{r} \sum_{p=0}^{m} \binom{r}{x} \binom{\binom{r-x}{2}}{m-p} f_{p+x+1,\, x} \ .$$

- One can show that $f_{n,r} = 0$ if $2r+1 < n$ – this implies that the F-matrix has about half the entries present in $a_{n,r}$ – this is exactly what we sought to achieve.
- Thus we have a sequence

$$p_d(n) \longrightarrow a_{n,r} \xrightarrow{\mu_*} c_{m,x} \xrightarrow{\sigma_2} d_{m,x} \xrightarrow{\sigma_3} \cdots \longrightarrow f_{n,r} \ .$$

- Using the F-matrix we have determined partitions of 25 in all dimensions.

So what is $p_{100}(25)$?

$$p_{100}(25) = 14\ 87812\ 28118\ 61642\ 06136\ 98833\ 39386$$

Thankfully, we did not have to compute it directly.
Credits: Arun K. Jayaraman (currently a graduate student in Physics at CMU) and Prof. Paul Bratley (who has been retired for 12 years but still enjoys writing code as he put it while sending me a very fast implementation of the Bratley-McKay algorithm).

$$F = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 & 3 \\ 1 & 7 \\ 1 & 11 & 16 \\ 1 & 18 & 58 \\ 1 & 26 & 135 & 125 \\ 1 & 38 & 293 & 618 \\ 1 & 52 & 574 & 1927 & 1296 \\ 1 & 73 & 1089 & 5256 & 8220 \\ 1 & 97 & 1960 & 12982 & 32380 & 16807 \\ 1 & 131 & 3468 & 30320 & 107270 & 131897 \\ 1 & 172 & 5955 & 67414 & 319530 & 633442 & 262144 \\ 1 & 227 & 10085 & 145045 & 888983 & 2490187 & 2483096 \end{pmatrix}$$

Part II

Asymptotics of higher-dimensional partitions

(Work done with Nicolas Destainville;

Srivatsan Balakrishnan and Naveen S. Prabhakar)

On the asymptotics of $p_d(n)$

- An important result due to Bhatia et. al. in 1997 states that
$$\lim_{n\to\infty} n^{-d/d+1} \log p_d(n) =: \widehat{\beta}_1^{(d)} \quad \text{(a constant)}$$
- In 2003, Mustonen and Rajesh used a Monte Carlo simulation and showed that for solid partitions $\widehat{\beta}_1^{(3)} = 1.79 \pm 0.01$ which is close the one given by MacMahon numbers i.e., $\frac{4}{3}[3\zeta(4)]^{1/4} \sim 1.78$.
- They conjectured that exact answer would the one given by the MacMahon numbers.
- Conjecture (Weak Form): The constant for d-dimensional partitions is the same as the constant appearing in the asymptotics of d-dimensional MacMahon numbers i.e.,

$$\widehat{\beta}_1^{(d)} = \frac{d+1}{d}\Big[d\,\zeta(d+1)\Big]^{\frac{1}{d+1}} .$$

A serendipitous discovery

- Along with some undergraduate students, I was enumerating the numbers of solid partitions extending work due to Knuth and Mustonen-Rajesh.
- In order to have good estimates of computer run times, I used a one-parameter formula to estimate $p_3(n)$ for $56 \leq n \leq 62$.
- This formula was based on the asymptotics of the MacMahon numbers $m_3(n)$. One has

$$\log m_3(n) \sim \tfrac{4}{3}[3\zeta(4)]^{1/4}\, n^{3/4} + \frac{\zeta(3)}{2[3\zeta(4)]^{1/2}}\, n^{1/2} - \frac{\zeta(3))^2}{8[3\zeta(4)]^{5/4}}\, n^{1/4} - \frac{61}{96}\log n + \cdots$$

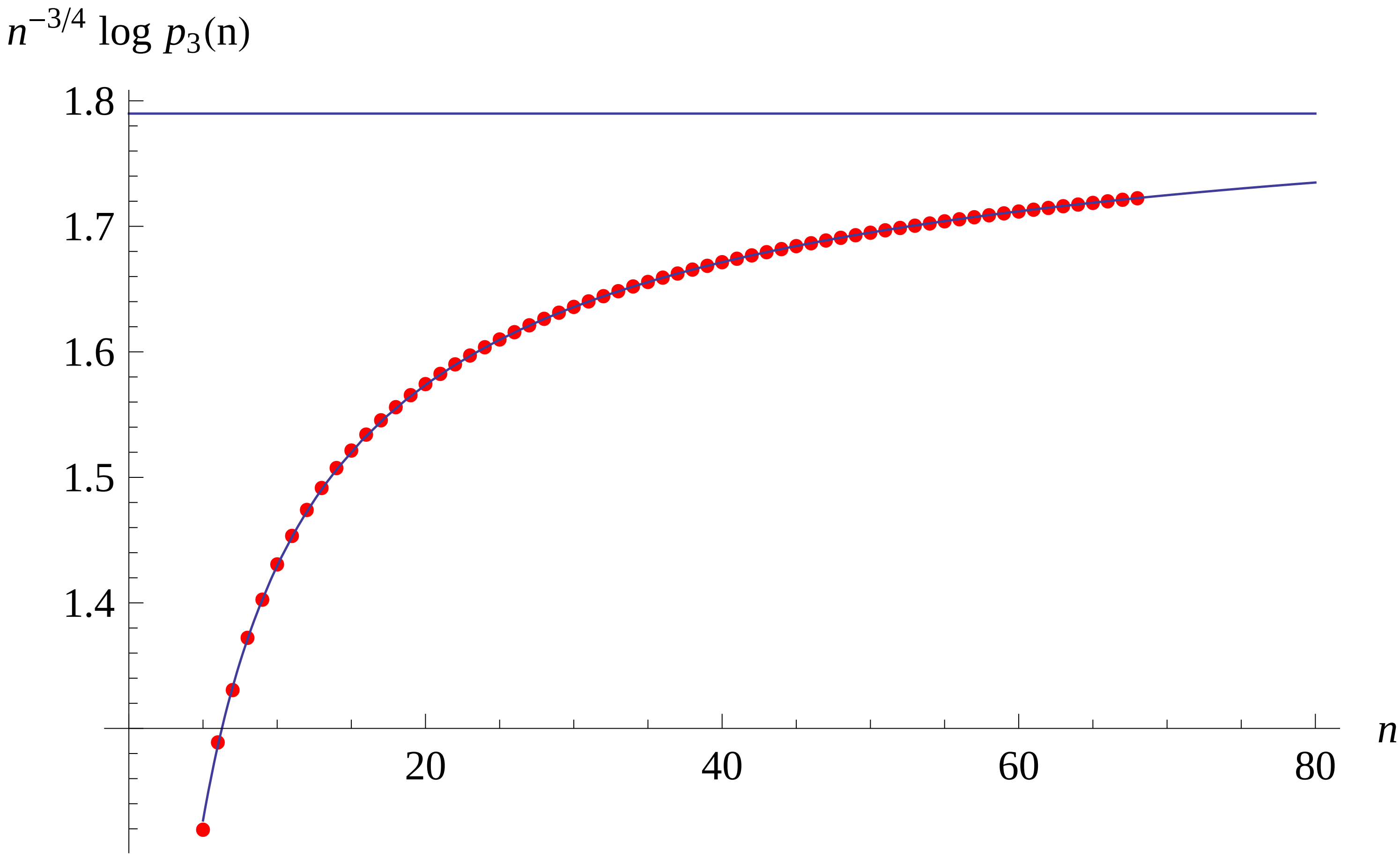
- So I used the above formula along with an overall multiplicative constant. This constant was fixed by required the exact answer for $p_3(55)$.
- Note that $n = 55$ is not large from an asymptotic viewpoint.

A serendipitous discovery

- In the table below, I list two numbers – the first one is the prediction and the second one (in green) is the exact answer obtained later.

55	used to fix constant 886033384475166	59	6521164672460061 6520649543912193
56	1464095834419295 1464009339299229	60	10684975704975763 10684614225715559
57	2414026269758682 2413804282801444	61	17472313806874724 17472947006257293
58	3971801006366828 3971409682633930	62	28514975666146341 28518691093388854

- The fact that 3-5 digits (shown in red) come out right is surprising and was the basis of a conjectures in weak form and strong form.



Plot of $n^{-3/4}\log p_3(n)$ for $n \in [5, 68]$ (red dots). The blue curve is the asymptotic formula normalized to give the correct answer for $n = 68$ and the horizontal line is the conjectured value for $n \to \infty$.

The conjecture in its strongest form

- Conjecture (Strong Form): The asymptotics of the d-dimensional partitions is identical to the asymptotics of the MacMahon numbers i.e.,

$$\log p_d(n) \sim \sum_{r=1}^{d} \beta_r^{(d)}\, n^{\frac{d-r+1}{d+1}} + \gamma^{(d)} \log n + \cdots \ .$$

- This is equivalent to requiring that the exponents, $a^{(d)}(n)$, for large enough n, go as

$$a^{(d)}(n) - \binom{n+d-2}{d-1} = \mathcal{O}(1)\ .$$

- Clearly there is a sequence of conjectures that are stronger than the weak conjecture but weaker than the above conjecture.

The conjectures are false!

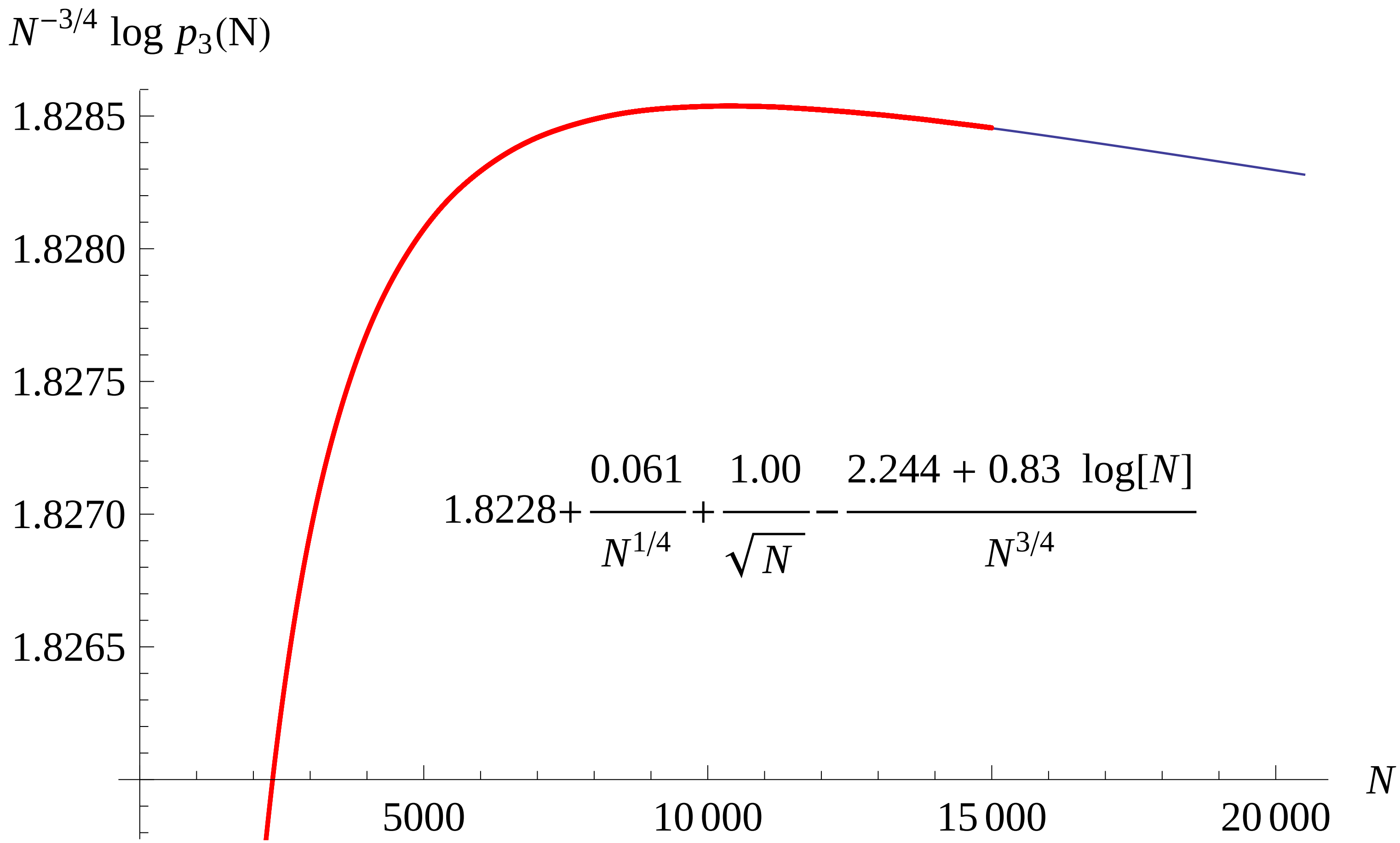
In collaboration with Nicolas Destainville, I set out to see if the conjectures are true. We ran Monte Carlo simulations for number of solid partitions going up to $n \sim 10000$. We then fit the data to the form of the asymptotic formula. We find

$$\lim_{n\to\infty} n^{-\frac{3}{4}} \log p_3(n) \sim 1.8228+0.061n^{-\frac{1}{4}}+1.00n^{-\frac{1}{2}}-(2.244+0.83\log n)n^{-\frac{3}{4}}$$

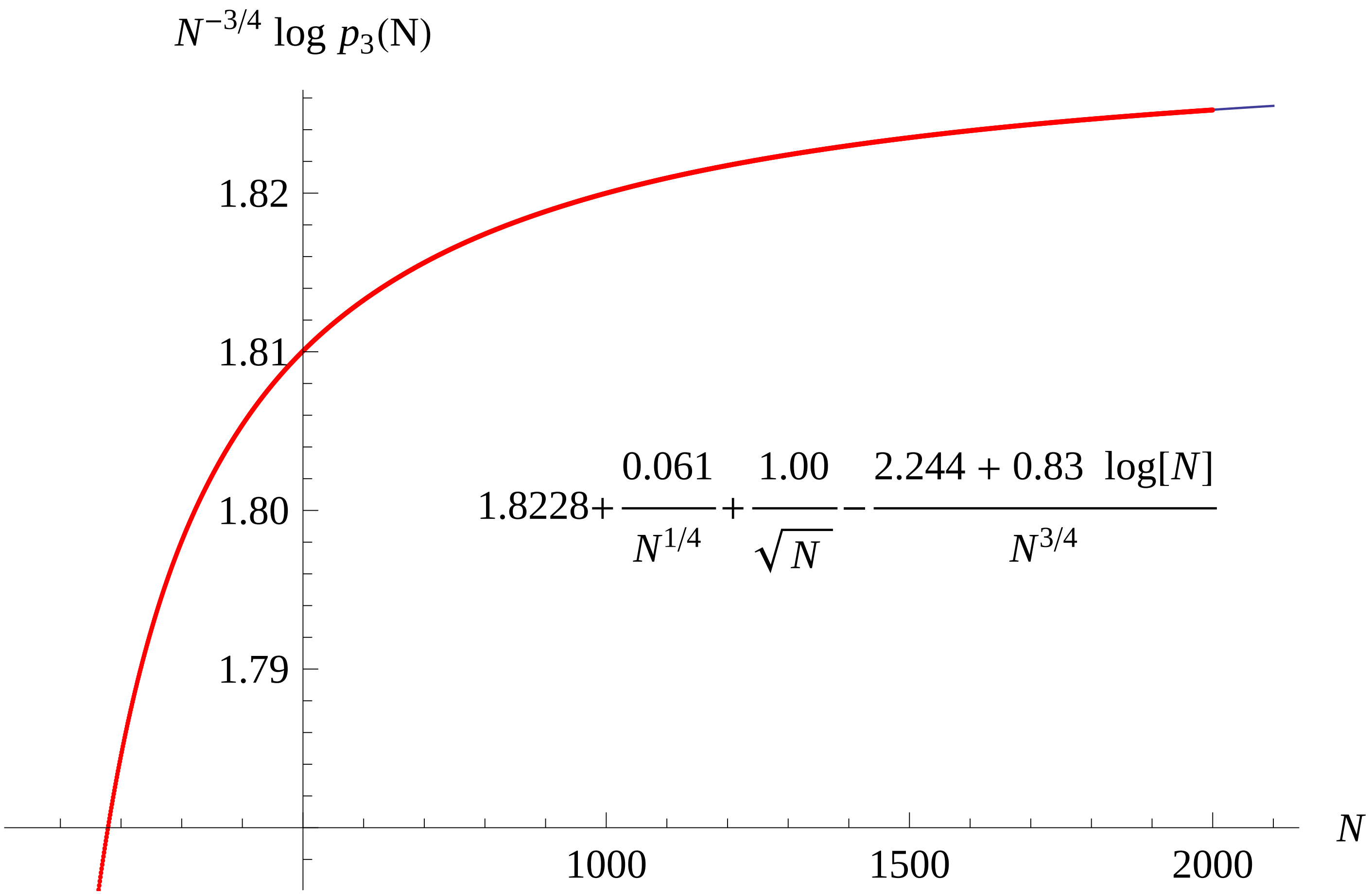
This has to be compared with the following asymptotic formula for the corresponding MacMahon numbers:

$$\lim_{n\to\infty} n^{-\frac{3}{4}} \log m_3(n) \sim 1.7898 + 0.3335n^{-\frac{1}{4}} - 0.04144n^{-\frac{1}{2}} + \cdots$$

Note that there is small deviation of the leading coefficients. This implies that the numbers of solid partitions exceeds the corresponding MacMahon number in contrast to what we see at low numbers.



Plot of $n^{-3/4}\log mc_3(n)$ for $n\in[50,15000]$ (red dots). The blue curve is the fit to the data in the range $[50,10000]$.



Plot of $n^{-3/4}\log mc_3(n)$ for $n \in [50, 2000]$ (red dots). The blue curve is the fit to the data in the range $[50, 10000]$.

Concluding Remarks

- We have seen how a sequences of transforms with their associated combinatorial interpretations helped us simplify the problem of enumerating partitions in any dimension to determining the F-matrix.
- What is lacking however is an algorithm that directly computes its entries. We suspect that it is unlikely since identifying whether a given FD is reducible and finding its reducible components is bound to be computationally intensive.
- We could however do something indirect – come up with closely related counting problems that have easily identifiable attributes. We hope to able to determine enough entries in the F-matrix to determine partitions of 30 with minor improvements.

Concluding remarks

- MacMahon computed tables of one-dim partitions up to $n = 200$ – these tables enabled Ramanujan to come up with his work on congruences. Maybe our exact enumeration will be the modern day analog of MacMahon's tables for higher-dimensional partitions.
- Our negative result on the asymptotics of solid partitions is not terrible. In fact, we find that the Monte Carlo data till about 300 works with the one-parameter formula implied by the conjecture. It is not clear as to why something changes at around 300.
- It also shows that the functional form is identical to that of the MacMahon numbers albeit with different coefficients.

$$p_{100}(24) = 99589\ 22039\ 36931\ 56439\ 87491\ 73261$$