The Gauge Orbit Space & Mass Gap

considerations from lower dimensions

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IMSc@50: Topology and Differential Geometry in Quantum Physics

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Chennai, INDIA

March 14, 2012

Why are YM(2), YM(3) important?

- Yang-Mills (2) or Yang-Mills (1+1):
  - No propagating degrees of freedom
  - Exactly solvable on any Riemann surface (Witten; Gross & Taylor)
- Yang-Mills (3) or Yang-Mills (2+1):
  - Super-renormalizable, no difficulties with running coupling constant, dimensional transmutation
  - Has propagating degrees of freedom
  - YM(3) describes the high temperature phase of real QCD
  - SUSY YM (2+1) and SUSY YM-CS relevant to M2 and D2 branes; e.g., D2 flow to CFT uses $\mathcal{N} = 8$.
  - May give some insights into YM (4)
- Yang-Mills (4) or Yang-Mills (3+1):
  - Most interesting and physically relevant case, but too difficult

- Perturbation theory
  - Spectacularly successful, but limited in kinematic regime
- Qualitative analyses of nonperturbative structure
  - Work in the 70s and early 80s clarified many general features such as confinement, chiral symmetry breaking (with matter added)
- Lattice gauge theory
  - Very good numerical results for many quantities (spectrum, some matrix elements, etc.)
  - Physics behind many features unclear
- Geometry of configuration space (FEYNMAN, SINGER, *etc.*)
  - This has promise as a complementary approach
  - We will explore this approach in this talk for the 2+1 dimensional theory

A Plan

- Generalities
- Feynman's attempt
- Volume for $\mathcal{A}/\mathcal{G}_*$
- Argument for the wave function
- Volume for $\mathcal{A}/\mathcal{G}_*$, 3-d considerations
- SUSY theories
  - $\mathcal{N} = 1$
  - $\mathcal{N} \geq 2$

- Consider $SU(N)$ Yang-Mills theory in 2+1 dimensions, i.e., on $\mathbb{R}^{2,1}$.
- The gauge potential (connection) A is of the form $A = (-it^a)A^a_\mu dx^\mu$ where t^a are hermitian $N \times N$ matrices forming a basis for the Lie algebra of $SU(N)$
- The action for the theory is

$$
\begin{aligned}
S &= -\frac{1}{2\,e^2}\int d^3x\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = \frac{1}{4\,e^2}\int d^3x\,F^a_{\mu\nu}F^{a\mu\nu}\\
F &= dA + A\wedge A = \frac{1}{2}(-i\,t^a)\,F^a_{\mu\nu}\,dx^\mu\wedge dx^\nu
\end{aligned}
$$

- e^2 is the coupling constant with the dimension of mass.
- We use a Hamiltonian analysis. Of the three components A_0, A_i, $(i = 1, 2)$, we can set $A_0 = 0$ as a gauge choice.
- The action and the Hamiltonian are then

$$
\begin{aligned}
S &= \int d^3x\,\frac{1}{2}\left[e^2\,\dot{A}^a\dot{A}^a - \frac{B^aB^a}{e^2}\right], \qquad B^a = F^a_{12}\\
\mathcal{H} &= \frac{1}{2}\int d^2x\,\left[e^2E^2 + \frac{B^2}{e^2}\right] = \int d^2x\,\left[-\frac{e^2}{2}\frac{\delta^2}{\delta A^a_i\,\delta A^a_i} + \frac{B^2}{2\,e^2}\right]
\end{aligned}
$$

- The statement about mass gap is that the spectrum of this Hamiltonian has a gap; the lowest excited state is separated from the vacuum by an energy gap.
- In perturbation theory (a fair approximation for modes of large momenta compared to e^2), the excitations are "gluons" with $E \sim p$. So a nonzero mass gap can be useful in excluding gluons from the spectrum.
- Define

$$\begin{aligned} \mathcal{A} &= \{\text{Space of all smooth gauge potentials (connections) on } \mathbb{R}^2\} \\ \mathcal{G}_* &= \{g(\vec{x}) : \mathbb{R}^2 \to SU(N),\ g \to 1 \text{ as } |\vec{x}| \to \infty\} \end{aligned}$$

- Gauge transformations act on A as

$$A \to A^g = g^{-1} A\, g + g^{-1} dg, \qquad g \in \mathcal{G}_*$$

- The physical configuration space (or gauge orbit space) is $\mathcal{C} = \mathcal{A}/\mathcal{G}_*$.
  - The kinetic term of the Hamiltonian is to be defined as a Laplacian on $\mathcal{C}$
  - Wave functions are functions on $\mathcal{C}$, because of the Gauss law

- As a fiber bundle, $(\mathcal{A}, \mathcal{G}_*, \mathcal{C})$ is nontrivial ($\exists$ Gribov problem)
- $\mathcal{A}$ is an affine space, but it is easy to see that $\mathcal{C}$ has very nontrivial topology and geometry.
- For example, $\Pi_2[\mathcal{C}] = \mathbb{Z}$. (There are many other nonzero Π_n, $n > 2$ as well.)
- Feynman (1981) suggested the reason for the mass gap is that the distance between any two points in $\mathcal{C}$ cannot become arbitrarily large.
- The wave functions (on $\mathcal{C}$) on which the Laplacian acts cannot have arbitrarily long wavelengths, so there should be a gap.
- Singer pointed out that this cannot be true.
- The physically relevant metric for $\mathcal{A}$ is given by the Euclidean distance. For $\mathcal{C}$, we define

$$s^2(A, A') = -Inf_g \int d^2x \, \mathrm{Tr}(A' - A^g)^2$$

- Consider, as an example, in $SU(2)$ gauge theory

$$
\begin{aligned}
A_n &= (-it^3)\, i\, n\, (z\,\bar{z})^{n-1} \frac{(z\, d\bar{z} - \bar{z}\, dz)}{[1 + (z\,\bar{z})^n]} \\
F_n &= (-it^3)\, (-4\, n^2) \frac{(z\,\bar{z})^{n-1}}{[1 + (z\,\bar{z})^n]^2}\, dx_1 \wedge dx_2 \qquad \text{(Nothing pathological)}
\end{aligned}
$$

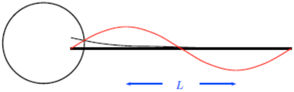
where $z = x_1 - i\, x_2,\ \bar{z} = x_1 + i\, x_2$.

- In this case,

$$s^2_{\mathcal{C}}(A, 0) = 8\pi n$$

For any value L^2, we can find an A, namely, A_n, with $n \geq (L^2/8\pi)$, for which $s^2(A, 0) > L^2$.

- These are the so-called "spikes" on $\mathcal{C}$.
- A long wavelength standing wave on such a spike can have arbitrarily low energy, seemingly vitiating Feynman's argument.



- May be not ! It could be similar to the 2-dim Schrödinger problem

$$\mathcal{H} = -\frac{\nabla^2}{2M} + \lambda(x^2 + x^2 y^2)$$

- The potential is zero along the y-axis, $(x = 0)$, and one can, *a priori*, think of long wavelength wave functions along this direction.
- The valley along the y-axis gets narrower as x becomes large. The zero-point energy of transverse directions (roughly, $\omega \sim \sqrt{1 + y^2}$) lifts the potential.
- Something similar could happen for YM, but we need a measure for the transverse directions.
- This can be done as follows

- Parametrization of potentials: $A_0 = 0$ and we use complex coordinates $z = x_1 - ix_2$ with

$$\tfrac{1}{2}(A_1 + iA_2) = -\partial M\, M^{-1}, \qquad \tfrac{1}{2}(A_1 - iA_2) = M^{\dagger -1}\bar{\partial} M^{\dagger}$$

$M \in SL(N, \mathbb{C})$, for gauge group $SU(N)$. (More generally, $G \Rightarrow G^{\mathbb{C}}$.)

- $H = M^{\dagger}\, M \in SL(N, \mathbb{C}/SU(N)$ is the basic gauge-invariant variable we need.
- The variation of the potentials is given by

$$\delta A = -D(\delta M M^{-1}) \qquad \delta\bar{A} = \bar{D}(M^{\dagger -1}\delta M^{\dagger})$$

- We then have

$$ds^2_{\mathcal{A}} = \int d^2x \,\mathrm{Tr}(\delta A \delta\bar{A}) = \int \mathrm{Tr}\left[(M^{\dagger -1}\delta M^{\dagger})(-\bar{D}D)(\delta M M^{-1})\right]$$

$$ds^2_{SL(N,\mathbf{C})} = \int \mathrm{Tr}(M^{\dagger -1}\delta M^{\dagger}\; \delta M M^{-1})$$

$$d\mu_{\mathcal{A}} = \det(-\bar{D}D)\; \underbrace{d\mu(M, M^{\dagger})}_{\text{Haar measure for } SL(N,\mathbb{C})}$$

- We can split the $SL(N, \mathbb{C})$ volume element as

$$d\mu(M, M^\dagger) = \underbrace{d\mu(H)}_{\text{Haar for } SL(N,\mathbb{C})/SU(N)} \qquad \underbrace{d\mu(U)}_{\text{Haar for } SU(N)}$$

$$d\mu_{\mathcal{A}} = \det(-\bar{D}D)\, d\mu(H)\;\; d\mu(U)$$

- For the gauge-orbit space

$$\begin{aligned} d\mu(\mathcal{C}) &= \det(-\bar{D}D)\, d\mu(H) \\ &= d\mu(H)\;\exp\left[2\, c_A\, S_{wzw}(H)\right] \end{aligned}$$

$S_{wzw}(H)$ is the Wess-Zumino-Witten (WZW) action,

$$S_{wzw}(H) = \frac{1}{2\pi}\int \mathrm{Tr}(\partial H \bar{\partial} H^{-1}) - \frac{i}{12\pi}\int \mathrm{Tr}(H^{-1}dH)^3$$

$c_A\;\delta_{ab} = f_{amn}f_{bmn} = N\,\delta_{ab}$ for $SU(N)$.

- The inner product for wave functions is then

$$\langle 1|2\rangle = \int d\mu(H) \exp[2\, c_A\, S_{wzw}(H)]\; \Psi_1^* \Psi_2$$

- Remarks:
  - The essential step is an anomaly calculation, so this result is robust, independent of the regulators used.
  - Integration with $d\mu(\mathcal{C})$ is equivalent to the calculation of correlators in the hermitian WZW model, so it can be done unambiguously (GAWEDZKI & KUPIAINEN)
  - The same result is obtained by taking a suitable limit in the exact solutions of the 2-dimensional YM theory (WITTEN; GROSS & TAYLOR; SENGUPTA; ASHTEKAR *et al*)
  - In terms of the magnetic field B,

$$d\mu(\mathcal{C}) \sim d\mu(H) \exp\left[-\frac{c_A}{2\pi} \int B \frac{1}{p^2} B + \ldots\right]$$

    showing a sharp cutoff for modes of small momenta p. This is the essence of the mass gap.

- The Hamiltonian and the wave functions can be expressed as functions of the (scaled version of the) current $J = (c_A/\pi)\,\partial H\, H^{-1}$.
- The Wilson loop operator is given by

$$W(C) = \mathrm{Tr}_R\, \mathcal{P} e^{-\oint_C A} = \mathrm{Tr}\,\mathcal{P}\exp\left(\frac{\pi}{c_A}\oint_C J\right)$$

All gauge-invariant quantities can be made from J.

- The Hamiltonian is given by

$$\mathcal{H} = m\left[\int_u J^a(\vec{u})\frac{\delta}{\delta J^a(\vec{u})} + \int_{u,v}\left(\frac{c_A}{\pi^2}\frac{\delta_{ab}}{(u-v)^2} - i\,\frac{f_{abc}\,J^c(\vec{v})}{\pi(u-v)}\right)\frac{\delta}{\delta J^a(\vec{u})}\frac{\delta}{\delta J^b(\vec{v})}\right]$$
$$+\ \frac{\pi}{m c_A}\int_u :\bar{\partial}J_a(u)\,\bar{\partial}J_a(u):$$

where $m = e^2 c_A/2\pi$.

An Aside on Regularization

All calculations have to be done with proper regularization.

- We start with a regularization of the δ-function

$$\delta^{(2)}(u,w) \Longrightarrow \sigma(\vec{u},\vec{w},\epsilon) = \frac{1}{\pi\epsilon}\exp\left(-\frac{|u-w|^2}{\epsilon}\right)$$

- This is equivalent to

$$\begin{aligned} \bar{G}(\vec{x},\vec{y}) &= \frac{1}{\pi(x-y)} \\ \Longrightarrow \quad \bar{\mathcal{G}}(\vec{x},\vec{y}) &= \int_u \bar{G}(\vec{x},\vec{u})\, \sigma(\vec{u},\vec{y};\epsilon) H(u,\bar{y}) H^{-1}(y,\bar{y}) \end{aligned}$$

- This simplifies as

$$\overline{\mathcal{G}}_{ma}(x,y) = \frac{1}{\pi(x-y)}\left[\delta_{ma} \; - \; e^{-\frac{(x-y)^2}{\epsilon}} \; [H(x,\bar{y})H^{-1}(y,\bar{y})]_{ma}\right]$$

All results checked using regularized expressions, with a single regulator from beginning to end.

- Absorb $\exp(2\,c_A S_{wzw})$ from the inner product into the wave function by $\Psi = e^{-c_A S_{wzw}(H)}\,\Phi$. The Hamiltonian acting on Φ is

$$\mathcal{H} \to e^{-c_A S_{wzw}(H)}\ \mathcal{H}\, e^{-c_A S_{wzw}(H)}$$

- Consider $H = e^{t^a \varphi^a} \approx 1 + t^a \varphi^a + \cdots$, a small φ limit appropriate for a (resummed) perturbation theory. The new Hamiltonian is

$$\mathcal{H} = \frac{1}{2} \int \left[-\frac{\delta^2}{\delta \phi^2} + \phi(-\nabla^2 + m^2)\phi + \cdots \right]$$

where $\phi_a(\vec{p}) = \sqrt{c_A\, p\bar{p}/(2\pi m)}\; \varphi_a(\vec{p})$.

- The vacuum wave function is

$$\Phi_0 \approx \exp\left[-\frac{1}{2} \int \phi^a \sqrt{m^2 - \nabla^2}\; \phi^a \right]$$

- Transforming back to Ψ,

$$\Psi_0 \approx \exp\left[-\frac{c_A}{\pi m}\int (\bar{\partial}\partial\varphi^a)\left[\frac{1}{m+\sqrt{m^2-\nabla^2}}\right](\bar{\partial}\partial\varphi^a) + \cdots\right]$$

- The full wave function must be a functional of J. The only form consistent with the above is

$$\Psi_0 = \exp\left[-\frac{2\pi^2}{e^2 c_A^2}\int \bar{\partial}J^a(x)\left[\frac{1}{m+\sqrt{m^2-\nabla^2}}\right]_{x,y} \bar{\partial}J^a(y) \; + \cdots\right]$$

since $J \approx (c_A/\pi)\partial\varphi + \mathcal{O}(\varphi^2)$.

- This indicates the robustness of the Gaussian term in Ψ_0, since this argument only presumes
    1. Existence of a regulator, so that the transformation $\Psi \Longleftrightarrow \Phi$ can be carried out
    2. The two-dimensional anomaly calculation

- For modes of low momenta, this has the form

$$\Psi_0 \approx \exp\left[-\int \frac{B^2}{4m\; e^2}\right]$$

- For the expectation value of the Wilson loop, this gives

$$\langle W(C)\rangle = \langle \mathrm{Tr}_R\, \mathcal{P} e^{-\oint_C A}\rangle = \exp\left(-\sigma_R\, \mathrm{Area}(C)\right), \qquad \sigma_R = e^4\, \frac{c_A c_R}{4\pi}$$

- This agrees, within 2%, with numerical simulations for $SU(2)$ to $SU(6)$ (12 representations), G_2 (8 representations) and to within 1% of the extrapolated large N limit. (Lucini & Teper; Bringoltz & Teper; Hari Dass & Majumdar; Kiskis & Narayanan; Welleghausen *et al*)
- This is nice, but not the main point for today.
- We want to relate the measure calculation to three-dimensional covariant Feynman diagram calculations, motivated by supersymmetric theories.

- For the calculation of the measure, the action is not important, so we can consider the Chern-Simons theory

$$S_{CS} = -\frac{k}{4\pi}\int \mathrm{Tr}\left[A\,dA + \frac{2}{3}A^3\right] = -\frac{k}{4\pi}\int d^3x\,\epsilon^{\mu\nu\alpha}\,\mathrm{Tr}\left[\left(A_\mu\partial_\nu A_\alpha + \frac{2}{3}A_\mu A_\nu A_\alpha\right)\right]$$

  We can take the level number $k = 0$ at the end.

- In a Hamiltonian quantization, the wave functions obey the Gauss law

$$\left[D\,\frac{\delta}{\delta A} - \frac{k}{2\pi}\bar{\partial}A - \sum_r (-i\,t^a)_{(r)}\,\delta^{(2)}(x - x_r)\right]\,\Psi = 0$$

- The state with no charges and the state with two (conjugate) charges are given by

$$\begin{aligned}\Psi_0 &= \chi_0\,\exp(k\,S_{wzw}(M))\\ \Psi_2 &= \chi(z_1, z_2)\,M(1)\,M^{-1}(2)\,\Psi_0\end{aligned}$$

- The normalizations are determined by

$$\begin{aligned} \mathcal{I}_0 &= |\chi_0|^2 \int d\mu(H)\ \exp[\bar{k}\, S_{wzw}(H)]\ = 1 \\ \mathcal{I}_2 &= |\chi(z_1, z_2)|^2 \int d\mu(H)\ \exp[\bar{k}\, S_{wzw}(H)]\ \ H(1)\, H(2)^{-1}\ = 1 \end{aligned}$$

- We know from previous result that $\bar{k} = k + 2\, c_A$, but for the moment, we will pretend that it is not known.
- The correlators are determined by the Schrödinger equation (identical to the Knizhnik-Zamolochdikov (KZ) equation) and shows that
  - $\chi(z_1, z_2)$ obeys the KZ equation for level k $SU(N)$ WZW model, i.e., with parameter $\kappa = k + c_A$
  - The H-correlators obey the same KZ equation with parameter $-\bar{k} + c_A$
  - The z-dependence of the $|\chi(z_1, z_2)|^2$ must be canceled by the H-correlator, so we get

$$-(k + c_A) = -\bar{k} + c_A, \qquad \text{or} \qquad \bar{k} = k + 2\, c_A$$

- Finally, we know that we can determine the χ's and Wilson loop expectation value by a covariant 3-dim calculation via the effective action for the CS theory.
- One-loop calculations (with no further renormalizations) give the shift $k \to k + c_A$ going from S_{CS} to the effective action.
- Combining everything, we can find the shift from the Feynman diagrams and then use the arguments above to fix the volume element.
- For supersymmetric theories, the known shifts from the Feynman diagrams are (PISARSKI & RAO; KAO, LEE & LEE)

$$k \to \begin{cases} k + c_A & \mathcal{N} = 0 \\ k + c_A/2 & \mathcal{N} = 1 \\ k & \mathcal{N} \geq 2 \end{cases}$$

- Correspondingly, the volume elements should be

$$d\mu(\mathcal{C}) = d\mu(H)\; \exp\left(\bar{k}\, S_{wzw}(H)\right)\; d\,[\text{Fermions}]$$

$$\bar{k} = \begin{cases} 2\,c_A & \mathcal{N}=0 \\ c_A & \mathcal{N}=1 \\ 0 & \mathcal{N}\geq 2 \end{cases}$$

- The expectations for SUSY YM are then:
  - We can have mass gap for $\mathcal{N}=0,1$, no gap expected for $\mathcal{N}\geq 2$
  - For $\mathcal{N}=1$, one has to add a CS term for the YM theory for consistency (because of the parity anomaly), so there is always a mass gap (WITTEN; ELLIOTT & MOORE)
  - For $\mathcal{N}=2$, one expects zero gap from other considerations, but a stable suspersymmetric vacuum may not exist. (SEIBERG & WITTEN; GOMIS & RUSSO; UNSAL; AHARONY *et al*)
  - For $\mathcal{N}=4$, there is no mass term with unbroken supersymmetry (SEIBERG & WITTEN)
  - $\mathcal{N}=8$ is expected to flow to a CFT (SEIBERG; AHARONY *et al*; HERZOG *et al*)

- We can further check these expectations by explicit calculations
- Consider a Majorana field $\Psi = (\psi, \psi^\dagger)$ in the adjoint representation of $SU(N)$.
- There are two ways to go to gauge-invariant variables:

Choice I

$$\begin{pmatrix} \chi^a \\ \chi^{a\dagger} \end{pmatrix} = \begin{pmatrix} (M^{-1})^{ab}\psi^b \\ (M^\dagger)^{ab}\psi^{b\dagger} \end{pmatrix}$$

Choice II

$$\begin{pmatrix} \chi^a \\ \chi^{a\dagger} \end{pmatrix} = \begin{pmatrix} (M^\dagger)^{ab}\psi^b \\ (M^{-1})^{ab}\psi^{b\dagger} \end{pmatrix}$$

where $M^{ab} = 2\,\mathrm{Tr}(t^a\, M\, t^b\, M^{-1})$ is the adjoint representative of M.

- Calculate the Jacobian of the transformation to the new variables χ by integrating small variations in M to obtain

$$[d\psi\, d\psi^\dagger] = [d\chi\, d\chi^\dagger]\ \exp\left(\pm c_A S_{wzw}(H)\right)$$

(Upper plus sign for Choice I, lower minus sign for Choice II.)

- The action is

$$S = \int \left[-\frac{1}{4\,e^2} F^a_{\mu\nu} F^{a\mu\nu} - \frac{i}{2e^2} \bar{\Psi}^a (\gamma^\mu D_\mu \Psi)^a - \frac{k}{4\pi} \operatorname{Tr} \left(A_\mu \partial_\nu A_\alpha - \frac{2}{3} A_\mu A_\nu A_\alpha \right) \epsilon^{\mu\nu\alpha} \right.$$
$$\left. + i\, e^2 \operatorname{Tr} \bar{\Psi} \Psi \right]$$

- The supercharges are given by

$$Q^\dagger = \int (i\, \Psi^\dagger \gamma^i \frac{\delta}{\delta A^i} + \frac{1}{e^2} \psi^\dagger B), \qquad Q = \int (i\, \gamma^i \Psi \frac{\delta}{\delta A^i} + \frac{1}{e^2} \psi B)$$

- Carrying out the change to H for the gauge fields, the supercharge becomes

$$Q = i \int_x \underbrace{\psi^{a\dagger}(x) M^{ab}(x)}_{\chi^{\dagger b}} \left[\int_y \mathcal{G}(x,y) p^b(y) + \frac{k}{4c_A} \bar{J}^b(x) \right] - \frac{1}{e^2} \frac{2\pi}{c_A} \int \underbrace{\psi^a (M^{\dagger -1})^{ab}}_{\chi^b} \; \bar{\partial} J^b$$

- This identifies Choice II as the proper change to gauge invariant variables for the fermions.

- The wave functions are of the form

$$\Xi = \exp\left(\frac{k}{2}\left[S_{wzw}(M^\dagger) - S_{wzw}(M) + S_{wzw}(H)\right]\right)\ \Phi(H, \chi, \chi^\dagger)$$

- Absorbing the exponential factor into the measure, the inner product for Φ's involves

$$d\mu = d\mu(H)\ \exp\left[(k + (2-n)c_A)\, S_{wzw}(H)\right]$$

where we know $n = 1$ from previous arguments, but leave it as n for now.

- We obtain the supercharges (in terms of the gauge-invariant variables) and the Hamiltonian is given by $\mathcal{H} = \frac{1}{2}\{Q, Q^\dagger\}$. The terms relevant for the mass are

$$\mathcal{H} = \frac{e^2}{4\pi}(k + 2\, c_A - n\, c_A) \int \left[J^a \frac{\delta}{\delta J^a} + \chi^{a\dagger}(H^{-1})^{ab}\chi^b + \cdots\right]$$

We see equality of boson and fermion masses with a value $\frac{1}{2}m = (e^2 c_A/4\pi)$.

- The action is

$$
\begin{aligned}
S_{YM} &= -\frac{1}{4\,e^2}\int F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\,e^2}\int D_\mu\phi^a_A D^\mu\phi^a_A + \frac{1}{2\,e^2}\int F^a_A F^a_A \\
&\quad - \frac{i}{2e^2}\int \bar{\psi}^a_I\,\gamma^\mu D_\mu\,\psi^a_I - \frac{i}{2\,e^2}\int \bar{\omega}^a\,\gamma^\mu D_\mu\,\omega^a - \frac{i}{2\,e^2}\int \epsilon_{ABC}\,\bar{\psi}^a_A\,\psi^b_B\,\phi^c_C f^{abc} \\
&\quad + \frac{i}{e^2}\int \bar{\psi}^a_A\,\omega^b\,\phi^a_A f^{abc} - \frac{1}{4\,e^2}\int f^{abc}f^{amn}\,\phi^b_B\,\phi^c_C\,\phi^m_B\,\phi^n_C \\
S_{CS} &= -\frac{k}{4\pi}\epsilon^{\mu\nu\rho}\int \mathrm{Tr}\,(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho) \\
&\quad + \frac{k}{8\,\pi}\int\left(-i\,\bar{\psi}^a_I\,\psi^a_I + i\,\bar{\omega}^a\,\omega^a + 2\,F^a_A\,\Phi^a_A - \frac{1}{3}f^{abc}\epsilon_{ABC}\,\phi^a_A\,\phi^b_B\,\phi^c_C\right)
\end{aligned}
$$

- Fermions are ω^a and ψ^a_A, $A = 1, 2, 3$, corresponding to an $SO(3)$ R-symmetry.
- Setting $\omega,\ \phi_1,\ \phi_2,\ \psi_3\ =\ 0 \Longrightarrow \mathcal{N} = 2$ theory

- The supercharge contains, among other terms,

$$Q = \int \left[i\, \psi^{a\dagger} \frac{\delta}{\delta A^a} + \epsilon_{IJK} \psi^a_J \left(\Pi^a_{\phi_K} + i\, \frac{k}{4\pi} \phi^a_K \right) - \omega^a \left(\Pi^a_{\phi_I} - i\, \frac{k}{4\pi} \phi^a_I \right) + \cdots \right]$$

- The choice of gauge-invariant fermion variables is Choice II for ψ^a; this follows from the term connecting it to A^a.
- Given this, ω^a must go with Choice I.
- So we get $-2\, c_A$ from two of the three ϕ's; the contribution of the third ϕ and ω cancel out. i.e., $n = 2$,

$$d\mu = d\mu(H)\; \exp\left[(k + (2 - n) c_A)\, S_{wzw}(H) \right] = d\mu(H)\; \exp\left[k\, S_{wzw}(H) \right] \to d\mu(H)$$

- Independently, we can check that the SUSY algebra

$$\{Q, Q^\dagger\} = 2\, \mathcal{H}, \qquad [Q, \mathcal{H}] = 0$$

requires $n = 2$.

Conclusion

- The geometry of the gauge orbit space seems to be at the heart of the mass gap question for Yang-Mills theories
- There are direct and indirect ways to get some understanding of the gauge orbit space
- The expectations for the mass gap based on the geometry of $\mathcal{C}$ are consistent with other arguments.
- Thanks to my collaborators:

  Dimitra Karabali; Chanju Kim; Abhishek Agarwal; Alexandr Yelnikov

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Thank You

Additional Slides

- There may be a useful generalization to 3+1 dimensions. Define

$$d\mu(\mathcal{C})_{3d} = \frac{[dA]}{vol(\mathcal{G}_*)} \,\exp\left(-\frac{1}{4\mu}\int F^2\right)\Bigg]_{\mu\to\infty}$$

- This leads to

$$\begin{aligned}
\int d\mu(\mathcal{C})_{3d} &= \int \frac{[dA]}{vol(\mathcal{G}_*)}\, e^{\left(-\frac{1}{4\mu}\int F^2\right)}\Bigg]_{\mu\to\infty} \\
&= \langle 0|\, e^{-\beta\mathcal{H}}\, |0\rangle\Big]_{\beta,\mu\to\infty} = \int d\mu(\mathcal{C})_{2d}\; \Psi_0^*\Psi_0\Big]_{\mu\to\infty} \\
&= \int d\mu(\mathcal{C})_{2d}\, \exp\left(-\frac{\pi}{2\mu^2 c_A}\int F^2\right) < \infty
\end{aligned}$$

- On the torus, the Hamiltonian has a part which is Laplacian for the zero modes,

$$A_z = M \left[\frac{i\pi\, a}{\mathrm{Im}\,\tau} \right] M^{-1} - \partial_z M\, M^{-1}$$

- As one of the torus directions becomes small, there is an accumulation of the eigenvalues of the Laplacian for the zero modes.
- This could be the signal for deconfinement.

- The space of gauge potentials has the bundle structure

$$
\begin{array}{ccc}
\mathcal{G}_* \rightarrow & \mathcal{A} & \\
& \downarrow & \\
& \mathcal{A}/\mathcal{G}_* &
\end{array}
$$

- This bundle is nontrivial. In particular, $\Pi_2(\mathcal{A}/\mathcal{G}_*) = \mathbb{Z}$ and $\Pi_n(\mathcal{A}) = 0$. There are noncontractible 2-spheres in $\mathcal{A}/\mathcal{G}_*$
- An example of such a configuration is

$$
\begin{aligned}
H &= \cosh 2f + \mathcal{J} \sinh 2f \\
f &= \frac{1}{2} \log \left(\frac{z\,\bar{z} + w\,\bar{w} + \mu^2}{z\,\bar{z} + w\,\bar{w}} \right)
\end{aligned}
$$

$w,\ \bar{w}$ are coordinates of the 2-sphere in $\mathcal{A}/\mathcal{G}_*$.

- The matrix $\mathcal{J}$ is given by

$$\mathcal{J} = \begin{pmatrix} z\,\bar{z} - w\,\bar{w} & 2\,\bar{w}\,z \\ 2\,w\,\bar{z} & w\,\bar{w} - z\,\bar{z} \end{pmatrix}$$

- $w = 0, z = 0$ is a singular point. Move singularity to another point by

$$H \to V\,H\,\bar{V}, \qquad\qquad \bar{V} = \exp\left[\sigma_3\left(\log\frac{\bar{z}}{\bar{z} - \bar{a}}\right)\right]$$

- $S_{wzw}(H)$ unchanged, finite. The volume element is insensitive to this problem.