From the fuzzy Torus
to
Matrix Quantum Mechanics.

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Comm. Math. Phys. and Preprint Napoli
DSF-NA/03/01

Chennay 2003

Suppose we want to discretize the functions on a two dimensional torus.

One option is to consider matrices with entries the values of the functions on a lattice of points.

Multiplication is just the product of the single elements

Very little information is carried by this approximation

We can try a **fuzzy** approximation

Torus: $x_1, x_2 \in [0,1]$

Functions on a torus:

$a(x) = \sum_{mn} a_{mn} e^{2\pi i m x_1} e^{2\pi i n x_2}$

It is impossible to truncate this sum at a finite level, since the product will produce higher harmonics

Define finite q-dimensional clock and shift matrices:

$$U_1 = \begin{pmatrix} 1 & & & & \\ & e^{\frac{2\pi i}{q}} & & & \\ & & e^{2\frac{2\pi i}{q}} & & \\ & & & \ddots & \\ & & & & e^{(q-1)\frac{2\pi i}{q}} \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix}.$$

$$a = \sum_{m,n=1}^{q} a_{mn} U_1^n U_2^m$$

The sum is finite because $U_1^q = U_2^q = \mathbb{I}$

Now the harmonics retained are finite, the space is *finite* dimensional and the products is consistent at a price $U_1 U_2 = e^{\frac{2\pi i}{q}} U_2 U_1$

Noncommutative Torus

$$a = \sum_{m,n=-\infty}^{\infty} a_{mn} U_1^n U_2^m$$

$$\boxed{U_1 U_2 = e^{2\pi i\theta} U_2 U_1}$$

Infinite dimensional algebra

The archetipical example of a Noncommutative Compact geometry

When θ irrational is not *Morita Equivalent* to any commutative space

The Noncommutative Tori for θ rational or irrational are "topologically" very different

In general θ and $\theta' = \frac{a\theta+b}{c\theta+d}$ $ad - bc = 1$ integers give rise to Morita equivalent tori

We can try to approximate the Noncommutative Torus with a sequence of finite matrix algebras generated by shifts and clocks with

$$U_1 = \begin{pmatrix} 1 & & & & \\ & e^{2\pi i \frac{p_n}{q_n}} & & & \\ & & e^{2\,2\pi i \frac{p}{q}} & & \\ & & & \ddots & \\ & & & & e^{(q_n-1)\,2\pi i \frac{p_n}{q_n}} \end{pmatrix}$$

$$\theta_n = \frac{p_n}{q_n} \to \theta$$

a sequence of rationals converging to θ

Is there a sequence of finite dimensional algebras (matrices) ed an embedding map such that the limit (i.e. the completed set of coherent sequences) is the noncommutative torus?

NO

The Noncommutative Torus is not an *Approximatively finite algebra*

For example the K_1 of an AF algebra is always trivial, the one of the NCTorus is not

This would not be a problem were it not for the fact that we want to approximate this noncommutative geometry (coming from string theory) with a Matrix Algebra

It is however possible Pimsner-Voiculescu, LLS to *embed* the noncommutative torus in an AF algebra.

An AF algebra is a succession of algebras and embeddings:

$$A_0 \overset{\rho_1}{\hookrightarrow} A_1 \overset{\rho_2}{\hookrightarrow} A_2 \overset{\rho_3}{\hookrightarrow} \cdots \overset{\rho_n}{\hookrightarrow} A_n \overset{\rho_{n+1}}{\hookrightarrow} \cdots$$

At each level the A_i are sums of the matrix algebras $\mathrm{M}_n(\mathbb{C})$ or their block subalgebras:

$A_1 = \bigoplus_{j=1}^{n_1} \mathrm{M}_{d_j^{(1)}}(\mathbb{C})$ and $A_2 = \bigoplus_{k=1}^{n_2} \mathrm{M}_{d_k^{(2)}}(\mathbb{C})$

but since

$\rho_1(A_1) \subset A_2$

$A_1 \cong \bigoplus_{k=1}^{n_2} \bigoplus_{j=1}^{n_1} N_{kj}\, \mathrm{M}_{d_j^{(1)}}(\mathbb{C})$

ex: $A_1 = \mathrm{M}_3 \oplus \mathrm{M}_2 = \begin{pmatrix} a & \\ & b \end{pmatrix}$, $A_2 = \mathrm{M}_{13}$, $\rho(A_1) = \begin{pmatrix} a & & & & \\ & a & & & \\ & & a & & \\ & & & b & \\ & & & & b \end{pmatrix}$

Note that it is the embedding which defines the limit: *Bratteli diagrams*

Embeddings for the noncommutative torus

Any irrational θ has a unique expansion as a continued fraction:

$$\theta = \lim_n \theta_n = \lim_n \frac{p_n}{q_n}$$

$$\theta_n = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{\ddots \; c_{n-1} + \cfrac{1}{c_n}}}}$$

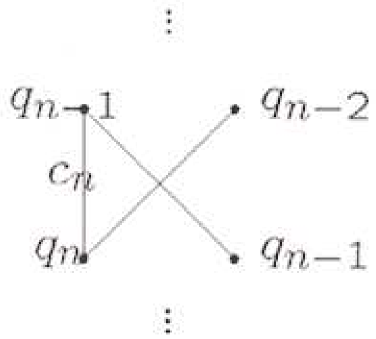
$$\begin{aligned} p_n &= c_n p_{n-1} + p_{n-2} \; , \; p_0 = c_0 \; , \; p_1 = c_0 c_1 + 1 \\ q_n &= c_n q_{n-1} + q_{n-2} \; , \; q_0 = 1 \; , \; q_1 = c_1 \end{aligned}$$

$A_n = \mathrm{M}_{q_n}(\mathbf{C}) \oplus \mathrm{M}_{q_{n-1}}(\mathbf{C})$

with the embeddings:

$$\begin{pmatrix} \mathcal{M} & \\ & \mathcal{N} \end{pmatrix} \xrightarrow{\rho_n} \begin{pmatrix} \left.\begin{matrix} \mathcal{M} & & \\ & \ddots & \\ & & \mathcal{M} \end{matrix}\right\} c_n & & \\ & \mathcal{N} & \\ & & \mathcal{M} \end{pmatrix}$$

The corresponding Bratteli diagram is:



Associated we have positive maps $\varphi_n : \mathbf{Z}^2 \to \mathbf{Z}^2$ defined by

$$\begin{pmatrix} q_n \\ q_{n-1} \end{pmatrix} = \varphi_n \begin{pmatrix} q_{n-1} \\ q_{n-2} \end{pmatrix} \quad , \quad \varphi_n = \begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix} .$$

The group $\mathrm{K}_0(A_\infty)$ is the inductive limit

$$\mathrm{K}_0(A_\infty) = \mathbf{Z} + \theta\mathbf{Z}$$

with $\mathrm{K}_0^+(A_\infty) = \left\{(z, w) \in \mathbf{Z}^2 \mid z + \theta w \geq 0\right\}$

At a finite level we can define the U's as before

$$U_1^{(n)} U_2^{(n)} = e^{2\pi i p_n / q_n} U_2^{(n)} U_1^{(n)}$$

That we are approximating the torus is given by the relation *(Pimsner–Voiculescu)*

$$\lim_{n\to\infty} \left\| \rho_n \left(U_a^{(n-1)} \oplus U_a^{(n-2)} \right) - U_a^{(n)} \oplus U_a^{(n-1)} \right\|_{A_n} = 0 \quad a = 1, 2$$

Note that $U_i^{(\infty)}$ is not a coherent sequence, it is a limit of coherent sequences, therefore $A_\theta \subset A_\infty$

The NCtorus in some sense is on the "boundary" of the algebra

An interesting corollary is the following:

It is a classic result in number theory that two irrational θ, θ' have the continued fraction expansion which is he same (up to a shift in the indices if and only if they are connected by the $SL(2, \mathbf{Z})$ transformation which defines Morita equivalence.

If A_θ and $A_{\theta'}$ are Morita equivalent, then θ and θ' have the same tail (up to a shift) in the continued fraction expansions, since in a Bratteli diagram what counts is only the infinite tail, the corresponding A_∞ are the same.

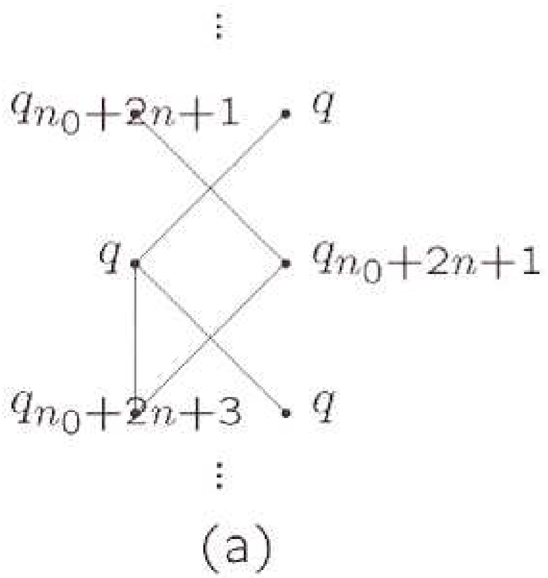
Morita equivalent noncommutative tori are subalgebra of the same AF algebra.

We also can define the Hilbert spaces: $\mathcal{H}_n = \mathbb{C}^{q_n} \oplus \mathbb{C}^{q_{n-1}}$ on which the algebra A_n naturally acts.

The embeddings are defined by:

$$\begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix} \xmapsto{\tilde{\rho}_n} \begin{pmatrix} \left.\begin{matrix} \frac{\vec{v}}{\sqrt{1+c_n}} \\ \vdots \\ \frac{\vec{v}}{\sqrt{1+c_n}} \end{matrix}\right\} c_n \\ \\ \vec{w} \\ \\ \frac{\vec{v}}{\sqrt{1+c_n}} \end{pmatrix}$$

The same reasoning can be done for the rational case (LLS), with the difference that this time the continued fraction will contain some zeros. The Bratteli diagram in this case is:



The AF algebras defined by (a) or (b) are the same.

So far we have mostly dealt with topology, but we can say something about geometry as well. We can in fact approximate also derivative operators.

There are two natural derivative operators on the noncommutative torus:

$$\delta_a(U_b) = 2\pi i\, \delta_{ab} U_b \quad , \quad a, b = 1, 2$$

from which we can define the connections

$$[\nabla_a, U_b] = 2\pi\, \delta_{ab} U_b \quad , \quad a, b = 1, 2$$

At the finite level there are the equivalent expression

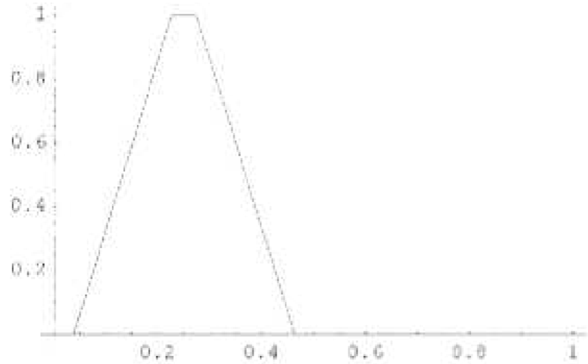
$$e^{-i\nabla_a^{(n)}}\, U_b^{(n)}\, e^{i\nabla_a^{(n)}} = e^{2\pi i \delta_{ab} r_a^{(n)}/q_n}\, U_b^{(n)}$$

with $r_a^{(n)}/q = R_a$, the "compactification radii"

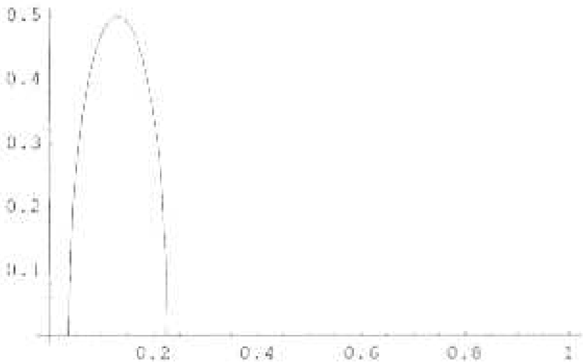
These (exponentiated) derivative operators have a well defined limit (LLS):

$$\lim_{n\to\infty} \left\| \rho_n \left(e^{i\nabla_a^{(n-1)}} \oplus e^{i\nabla_a^{(n-2)}} \right) - e^{i\nabla_a^{(n)}} \oplus e^{i\nabla_a^{(n-1)}} \right\|_{A_n} = 0$$

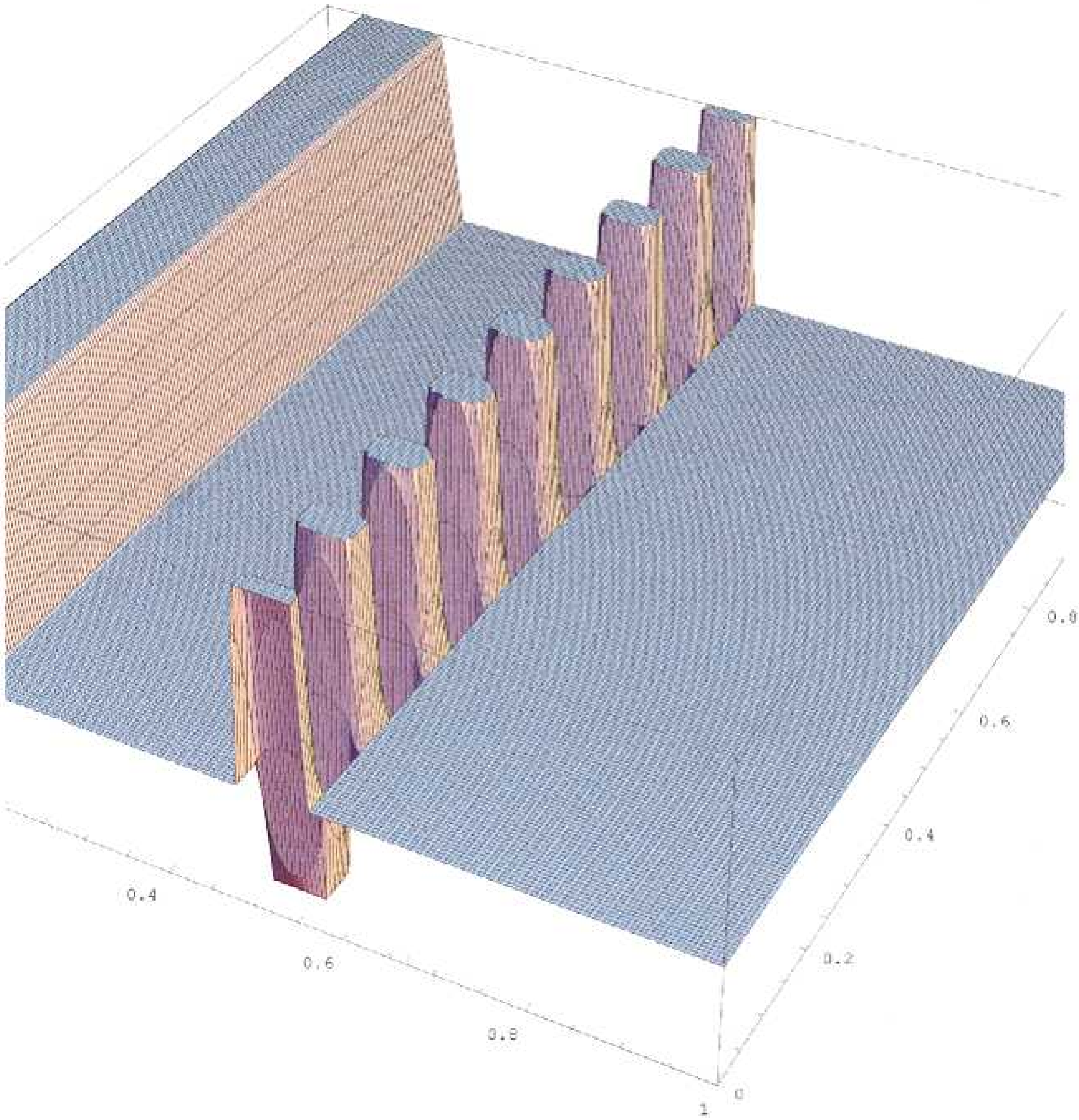
```
Plot[f[ϕ], {ϕ, 0, 1}]
Plot[g[ϕ], {ϕ, 0, 1}]
```



- Graphics -



- Graphics -


0.4
0.6
0.8
1
0
0.2
0.4
0.6
0.8

There is another matrix approximation in which the basis is made of solitons (Elliott and Evans)

Define the projectors on the two torus (Generalized Powers Rieffel)

$$
\begin{aligned}
\mathsf{P}_n &= V^{-q_{2n+1}}\, g_n(U) + f_n(U) + g_n(U)\, V^{q_{2n+1}} \,, \\
\mathsf{P}'_n &= U^{-q_{2n}}\, g'_n(V) + f'_n(V) + g'_n(V)\, U^{q_{2n}}
\end{aligned}
$$

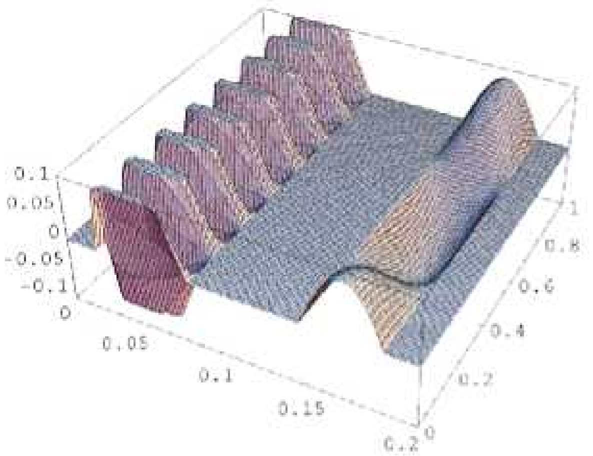
From now on use U, V instead of U_1, U_2, q, q' for q_{2n}, q_{2n+1}

f and g seen as functions of φ and φ' via the inverse of a Weyl map are bump functions

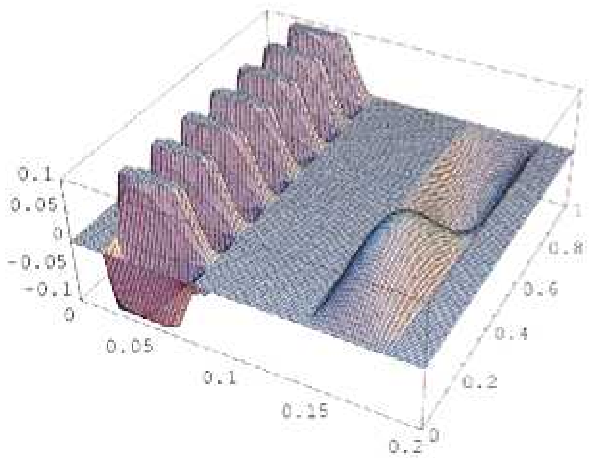
Consider also the translated projectors:

$$
\mathsf{P}^{ii} = V^{-q'}\, g\left(e^{2\pi \mathrm{i}\,(i-1)p/q}\, U\right) + f\left(e^{2\pi \mathrm{i}\,(i-1)p/q}\, U\right) + g\left(e^{2\pi \mathrm{i}\,(i-1)p/q}\, U\right) V^{q'}
$$

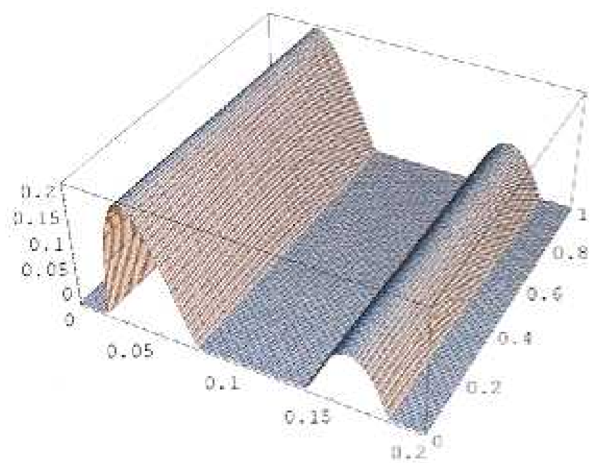
```
In[19]:= Plot3D[Re[p21[ϕ, ϕϕ]], {ϕ, 0, .2}, {ϕϕ, 0, 1}, PlotPoints → 100]
         Plot3D[Im[p21[ϕ, ϕϕ]], {ϕ, 0, .2}, {ϕϕ, 0, 1}, PlotPoints → 100]
         Plot3D[Abs[p21[ϕ, ϕϕ]], {ϕ, 0, .2}, {ϕϕ, 0, 1}, PlotPoints → 100]
```



Out[19]= - SurfaceGraphics -



Out[20]= - SurfaceGraphics -



Out[21]= - SurfaceGraphics -

P^{ii}, $i = 1, \ldots, q$ form a system of mutually orthogonal projection operators

$$\mathsf{P}^{ii}\,\mathsf{P}^{jj} = \delta^{ij}\,\mathsf{P}^{jj}$$

Let $\mathcal{H}_i \subset \mathcal{H}$ be the range of the projector P^{ii}

On each $\mathcal{H}_i$ the projector P^{ii} acts as the identity

To go from one space to the other we can use the partial isometric part of

$$\Pi^{i+1\,i} = \mathsf{P}^{i+1\,i+1}\,V\,\mathsf{P}^{ii}$$

which we call $P^{i+1\,i}$

fill more slots using

$$\mathrm{p}^{ij\dagger} = \mathrm{p}^{ji}$$

for the cases defined it results

$$\mathrm{p}^{ij}\mathrm{p}^{kl} = \delta^{jk}\mathrm{p}^{il}$$

using above as definition it is possible to fill the q^2 slots

except that...

$$\mathrm{p}^{1q} = \mathrm{p}^{12}\,\mathrm{p}^{23}\ldots\mathrm{p}^{q-1,q}$$

On the other hand, we could also define P^{1q} shifting $q-1$ times P^{12}

Where these two quantities equal we would close a matrix algebra

Instead they are equal up to a partial isometry z which is a unitary $\mathcal{H}_q \to \mathcal{H}_1$

The P^{ij} along with z close a subalgebra of the Noncommutative Torus

This is the algebra of matrix valued functions:

$$A(z) = \sum_{m=-\infty}^{\infty} \sum_{i,j=1}^{q} A_{ij,m}\, z^{m}$$

We can do an analogous construction exchanging U with V to obtain some $P^{ij'}$ and z' to obtain a subalgebra

$$\mathcal{A}_n \cong \mathbf{M}_{q_{2n}}\left(\mathbf{S}^1\right) \oplus \mathbf{M}_{q_{2n+1}}\left(\mathbf{S}^1\right)$$

The inductive limit of these algebras is the noncommutative torus

It is possible to approximate any element of the NCtours with an element of $\mathcal{A}_n$ substituting to U, V in the expansion

$$
\begin{aligned}
\mathsf{U}_n &= \left(\sum_{i=1}^{q_{2n}} (\omega_n)^{i-1}\, \mathsf{P}_n^{ii}\right) \oplus \left(\sum_{i'=1}^{q_{2n+1}-1} \mathsf{P}'^{\,i',i'+1}_n + z'\, \mathsf{P}'^{\,q_{2n+1},1}_n\right) \\
&= \begin{pmatrix} \mathcal{C}_{q_{2n}} & \\ & \mathcal{S}_{q_{2n+1}}(z') \end{pmatrix} , \\
\mathsf{V}_n &= \left(\sum_{i=1}^{q_{2n}-1} \mathsf{P}_n^{i,i+1} + z\, \mathsf{P}_n^{q_{2n},1}\right) \oplus \left(\sum_{i'=1}^{q_{2n+1}} (\omega'_n)^{i'-1}\, \mathsf{P}'^{\,i'i'}_n\right) \\
&= \begin{pmatrix} \mathcal{S}_{q_{2n}}(z) & \\ & \mathcal{C}_{q_{2n+1}} \end{pmatrix} ,
\end{aligned}
$$

$$
\mathcal{C}_q = \begin{pmatrix} 1 & & & & \\ & e^{2\pi \mathrm{i}\, p/q} & & & \\ & & e^{4\pi \mathrm{i}\, p/q} & & \\ & & & \ddots & \\ & & & & e^{2\pi \mathrm{i}\, p(q-1)/q} \end{pmatrix} \quad \text{with} \quad (\mathcal{C}_q)^q = \mathrm{I}_q
$$

$$
\mathcal{S}_q(z) = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ z & & & & 0 \end{pmatrix} \quad \text{with} \quad \left(\mathcal{S}_q(z)\right)^q = z\, \mathrm{I}_q \ .
$$

Given any element a of the NCtorus with expansion

$$a = \sum_{m,r=-\infty}^{\infty} a_{mr}\ U^m V^r$$

we associate to it the element

$$\Gamma_n(a) = \sum_{m,r=-\infty}^{\infty} a_{mr}\ (\mathsf{U}_n)^m (\mathsf{V}_n)^r$$

$$||a - \Gamma_n(a)|| \to 0$$

It is then possible to define integrals (trace) and derivatives in terms solely of matrix quantities. For example the integral:

$$\int \mathsf{a}_n(z, z') = \beta_{2n} \int_0^{\tau_n} d\tau\ \ \mathrm{Tr}\ \mathsf{a}_n(\tau) + \beta_{2n+1} \int_0^{\tau'_n} d\tau'\ \ \mathrm{Tr}'\, \mathsf{a}'_n(\tau')$$

$$\beta_{2n} = q_{2n}(\theta - \theta_n)$$

CONCLUSIONS

The Noncommutative Torus is not only an important example of noncommutative geometry, it may also have important physical applications in string/brane theory

The approximation presented here show that it is possible to describe (in the appropriate limit) a system of strings and branes as a quantum mechanical theory of matrices with values on two circles

But this would be another seminar

This is made possible by abandoning totally the deformation vision of the NCtorus *(still to some extent pointlike)* and instead considering a "nonlocal" basis whose elements are solitons and isometries