Quantum Hall effect and noncommutative field theories

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Basic QHE

2d electrons in strong magnetic field

- single particle

$$L = \frac{m}{2}\dot{\vec{x}}^2 + \vec{A}\cdot\dot{\vec{x}} - \frac{1}{2}\omega x^2 \qquad \left(A_i = \frac{B}{2}\epsilon_{ij}x_j\right)$$

inter-Landau energy splitting $\sim \frac{B}{m}$

$m \to 0$ LLL

$$[x_1, x_2] = \frac{i}{B}$$

2d configuration space = phase space of 1d system

$$H = \frac{\omega}{B}\left(a^\dagger a + \frac{1}{2}\right) \qquad a = \sqrt{\frac{B}{2}}\,(x_1 + i x_2)$$

eigenstates $\qquad |n\rangle = \frac{1}{n!}(a^\dagger)^n |0\rangle$

coherent state wavefunctions

$$\langle z|n\rangle \sim z^n e^{-|z|^2/2}$$

- many-particle

filling fraction $= \nu = \dfrac{\#\text{electrons}}{\#\text{states}} = \dfrac{2\pi\rho}{B}$

ρ = electron density $\qquad \dfrac{2\pi}{B} \sim$ area of flux quantum

IQHE $\qquad \nu = 1$

$$\Psi = \prod_{i<j} (z_i - z_j)\, e^{-\sum_i \frac{|z_i|^2}{2}}$$

circular droplet, uniform density $\rho = \dfrac{B}{2\pi}$

incompressibility $\sim$ gap

FQHE $\qquad \nu = \dfrac{1}{2p+1}$ $\qquad$ Coulomb interactions important

$$\Psi_{\text{Laughlin}} = \prod_{i<j} (z_i - z_j)^{2p+1}\, e^{-\frac{|z_i|^2}{2}}$$

circular droplet, uniform density $\rho = \dfrac{B}{2\pi(2p+1)}$

Excitations

$$\Psi = \underbrace{P(z_1 \ldots z_N)}_{\text{symmetric pol.}}\, \Psi_{\text{Laughlin}}$$

- edge excitations
- quasihole excitations $\qquad \prod_i (z_i - z_0)\, \Psi_{\text{Laughlin}}$

$$e^* = \frac{e}{2p+1}$$

Susskind's proposal

$$S = \int dt \, \frac{B}{2} \, \mathrm{Tr} \left\{ \epsilon_{ab} \left(\dot{X}_a + i [A_0, X_a] \right) X_b + 2\theta A_0 \right\}$$

X_1, X_2, A_0 are hermitian matrices

- eq. for A_0, Gauss law constraint

  $$[X_1, X_2] = i\theta$$

- $2\pi\theta$ = unit area occupied by electron

  $$\rho = \frac{1}{2\pi\theta} \underset{\left(\nu = \frac{2\pi\rho}{B}\right)}{\rightsquigarrow} \nu = \frac{1}{B\theta} \quad \text{(classically)}$$

noncommutativity matrix model $\longrightarrow$ discreteness for individual electrons

microscopic description of QHE ?

- connection to noncommutative $U(1)$ CS

  $$X^i = x^i + \theta \epsilon^{ij} A_j$$

  $$S_{NC} = \frac{1}{4\pi\nu} \int dt \, 2\pi\theta \, \mathrm{Tr} \left(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho \right) \epsilon^{\mu\nu\rho}$$

  $\theta \to 0$ $\quad U(1)$ CS : long distance behavior of QHE, fluid description

- <u>Drawback</u>

  X's are ∞-matrices (∞ many electrons)

  unique classical solution

  excitations $\rightsquigarrow$ introduce sources

<u>Regularization</u> (Polychronakos)

X^a : $N\times N$ matrices

$$S = \int dt\, \frac{B}{2} \mathrm{Tr}\left\{ \epsilon^{ab}\left(\dot{X}_a + i[A_0, X_a]\right)X_b + 2\theta A_0 - \omega X_a^2 \right\} + \Psi^\dagger\left(i\dot{\Psi} - A_0\Psi\right)$$

Ψ : complex N-vector

$$X_a \to U X_a U^{-1} \qquad \Psi \to U\Psi$$

Gauss law constraint ($A_0 = 0$ gauge)

$$G = -iB[X_1, X_2] + \Psi\Psi^\dagger - B\theta = 0$$

take trace

- $$\Psi^\dagger\Psi = NB\theta$$

$$\dot{\Psi} = 0 \quad \Rightarrow \quad \Psi = \sqrt{NB\theta}\,|v\rangle$$

$$|v\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

traceless part

- $$[X_1, X_2] = i\theta\left(1 - N|v\rangle\langle v|\right)$$

$$i\theta\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & -N+1 \end{pmatrix}$$

Equations for X_a

$$\dot{X}_a + \omega\,\epsilon_{ab} X_b = 0$$

solution : $$X_1 + iX_2 = e^{i\omega t} A$$

$$[A, A^\dagger] = 2\theta\left(1 - N|v\rangle\langle v|\right) \qquad \text{(constraint)}$$

Interesting classical solutions

- ground state (minimizes potential + constraint)

$$A = \sqrt{2\theta} \sum_{n=0}^{N-1} \sqrt{n}\, |n-1\rangle\langle n| \qquad |v\rangle = |N-1\rangle$$

$$R^2 = X_1^2 + X_2^2 = \sum_{n=0}^{N-2} \theta(2n+1)\, |n\rangle\langle n| + \theta(N-1)\, |N-1\rangle\langle N-1|$$

eigenvalues evenly distributed ~ circular droplet

area ~ $\pi\, 2N\theta$ $\Rightarrow$ density $= \frac{N}{\text{area}} = \frac{1}{2\pi\theta}$

- excitations

edge excitations $\left(A' = A + \sum_{n=0}^{N-1} \epsilon_n (A^\dagger)^n \right)$

quasihole excitations (charge q)

$$A = \sqrt{2\theta} \left(\sqrt{q}\, |N\rangle\langle 0| + \sum_{n=1}^{N-1} \sqrt{n+q}\, |n-1\rangle\langle n| \right)$$

$$R_n^2 = 2\theta(n+q) \qquad n = 0, 1, \ldots$$

circular droplet with a hole of area $2\pi\theta q$ at the origin

Quantization

$$[\psi_i, \psi_j^+] = \delta_{ij}$$

$$[(X_1)_{ij}, (X_2)_{kl}] = \frac{i}{B}\delta_{il}\delta_{jk}$$

Hamiltonian : $H = \omega\left(\frac{N^2}{2} + \sum A^+_{ij} A_{ji}\right)$

$$A = \sqrt{\frac{B}{2}}\,(X_1 + iX_2)$$

constraint: G = generator of unitary rotations

$$(\psi_i^+\psi_i - NB\theta)\,|\text{state}\rangle = 0 \quad \Rightarrow NB\theta = \text{integer}$$
$= \#$ of ψ oscillators

$$(G^a_X + G^a_\psi)\,|\text{state}\rangle = 0$$

adjoint (G^a_X) fundamental (G^a_ψ)

Write down $SU(N)$ singlets made out of A^+, ψ^+

$$(\mathrm{Tr}\, A^{+i})^{c_i}$$

$$\left[\epsilon^{i_1 \dots i_N}\,\psi^+_{i_1}(\psi^+A^+)_{i_2}(\psi^+A^{+2})_{i_3}\dots(\psi^+A^{+N-1})_{i_N}\right]^{\ell}$$

$$\Rightarrow B\theta = \ell = \text{integer} = \nu_{cl}^{-1} \quad \left(\nu = \frac{1}{\ell+1}\right)$$

(Nair, Polychronakos : $4\pi \times$ coef. of NC CS = integer
$\Rightarrow B\theta$ = integer)

Bak, Lee, Park

$\ell = 2p \longrightarrow$ fermions

$\ell = 2p+1 \longrightarrow$ bosons

$\left(\nu = \frac{1}{\ell+1}\right)$

Consider $\ell = 2p \rightsquigarrow 2pN$ ψ's

$$|\Psi\rangle_{ground} = \left[\epsilon^{i_1 \dots i_N} \psi^{\dagger}_{i_1} (\psi^{\dagger} A^{\dagger})_{i_2} \cdots (\psi^{\dagger} A^{\dagger N-1})_{i_N}\right]^{2p} |0\rangle$$

$$|\Psi\rangle_{excited} = \prod_{i=1}^{N-1} (\mathrm{Tr}\, A^{\dagger i})^{c_i} \; |\Psi\rangle_{ground}$$

$\psi^{\dagger} \to 1$

$A^{\dagger}_{ij} \to \delta_{ij} a^{\dagger}_j$

(Hellerman, Van Raamsdonk)

$$|\Psi\rangle_{matrix} \xrightarrow{\text{one-to-one}} |\Psi\rangle_{Laughlin}$$

same energy

same degeneracy at each energy level

Matrix model $\overset{?}{=}$ QHE

Introduce phase space coordinates via coherent state representation.

Two natural choices:

- $\underline{\text{A-representation}}$: diagonalize $A\ (= X_1 + iX_2)$

  eigenvalues of A = coordinates Z (phase space)

- $\underline{\text{X-representation}}$: diagonalize X_1

  eigenvalues of X_1 = 1d coordinates

$\underline{\text{A-representation}}$

$$\hat{A}_{ij} | Z, \phi \rangle = Z_{ij} \, | Z, \phi \rangle$$

$$\hat{\psi}_i \, | Z, \phi \rangle = \phi_i \, | Z, \phi \rangle$$

$$\langle \text{state} | \text{state} \rangle = \int dZ_{ij} \, dZ^*_{ij} \, d\phi_e \, d\bar{\phi}_e \, e^{-\mathrm{Tr}\, Z^\dagger Z} \, e^{-\bar{\phi}\phi} \, \langle \text{state} | Z, \phi \rangle \langle Z, \phi | \text{state} \rangle$$

integrate over ϕ's and nondiagonal Z_{ij}

$$= \int d^2 z_1 \ldots d^2 z_N \, \underbrace{P(z_1, \bar{z}_1, \ldots z_N, \bar{z}_N)}_{\text{probability distribution } = \psi^* \psi}$$

$\rightarrow$ identify wavefunction ψ

Find probability distribution for |ground state⟩

$$|2p+1\rangle = \left[\epsilon^{i_1 \dots i_N} \psi^\dagger_{i_1} (\psi^\dagger A^\dagger)_{i_2} \cdots (\psi^\dagger A^{\dagger\, N-1})_{i_N}\right]^{2p} |0\rangle$$

$$\left(\nu = \frac{1}{2p+1}, \quad 2p = B\theta\right) \qquad (NB\theta = \#\ \psi'_{osc})$$

Example $\quad p=0 \rightarrow \theta = 0$

$$\langle p=0|p=0\rangle = \underbrace{\int dz_{ij}\, dz^*_{ij}\, e^{-\mathrm{Tr}\, Z^\dagger Z}}_{\text{complex random matrix}} \underbrace{\int d\phi\, d\bar\phi\, e^{-\bar\phi \phi}}_{\text{trivial}}$$

$$\underset{\text{(Ginibre)}}{=} \int d^2 z_1 \dots d^2 z_N\, e^{-\sum |z_i|^2} \prod_{i<j} |z_i - z_j|^2$$

$$\Rightarrow \Psi(z_1 \dots z_N) = \nu = 1 \text{ Laughlin wave function}$$

$$= \prod_{i<j} (z_i - z_j)\, e^{-\frac{1}{2}\sum |z_i|^2}$$

(Brezin, Itzykson, Parisi, Zuber

$\theta = 0$ matrix model = 1d fermions)

<u>$p \neq 0$</u> $\quad \nu = \frac{1}{2p+1}$

Diagonalize $\quad Z \quad : \qquad Z = X E X^{-1}$

$E_{ij} = z_i \, \delta_{ij}$

$X = U \; Y \; V$ — diagonal with real positive elements

unitary

triangular
$Y_{ij} = 0 \quad i > j$ and $Y_{ii} = 1$
$\det Y = 1$

- $$\langle 2p+1 | Z, \phi \rangle = \left[\epsilon^{i_1 \dots i_N} \, \phi_{i_1} (Z\phi)_{i_2} \cdots (Z^{N-1}\phi)_{i_N} \right]^{2p}$$

$$= \left[\det(UY) \right]^{2p} \prod_{i<j} (z_i - z_j)^{2p} \prod_{i=1}^{N} \left[(UY)^{-1} \phi \right]_i^{2p}$$

- $$\mathrm{Tr}\, Z^{+} Z = \mathrm{Tr} \left(E^{*} H E H^{-1} \right) \qquad H = Y^{+} Y$$

- $$\int d\phi \, d\bar{\phi} \; e^{-\bar{\phi}\phi} \left[(UY)^{-1} \phi \right]_i^{2p} \overline{\left[(UY^{*})^{-1} \phi \right]_i^{2p}}$$

$$= \int d\bar{\xi} \, d\xi \; e^{-\bar{\xi} H \xi} \left(\xi_i \bar{\xi}_i \right)^{2p}$$

- $$dZ_{ij} \, dZ^{*}_{ij} = dz_i \, d\bar{z}_i \prod_{i<j} |z_i - z_j|^4 \prod_{i \neq j} dH_{ij}$$

Put everything together

$$\langle 2p+1|2p+1\rangle = \int d^2 z_i \prod_{i<j} |z_i - z_j|^{4p+4} \, dH_{ij} \, e^{-\mathrm{Tr}(E^* H E H^{-1})} \, d\bar{\xi}_i d\xi_i \prod_i (\bar{\xi}_i \xi_i)^{2p} e^{-\bar{\xi} H \xi}$$

<u>Simple case</u> $p=1 \Rightarrow \nu = \frac{1}{3}$, $N=2$

$$\langle p=1|p=1\rangle = \int dz_i \, d\bar{z}_i \; e^{-|z_1|^2 - |z_2|^2} \underbrace{|z_1 - z_2|^6 \left[1 + \frac{6}{|z_1 - z_2|^2} + \frac{12}{|z_1 - z_2|^4} \right]}_{\text{Probability distribution}}$$

- long distance $\sim$ Laughlin $\nu = \frac{1}{3}$
  short distance $\sim$ Laughlin $\nu = 1$

- factorizability
  $P = \psi^* \psi \Rightarrow \psi$ cannot be simultaneously antisymmetric and holomorphic

Similar features for arbitrary N

Find dependence on $|z_{N-1} - z_N|$

$$\text{Prob. distr.} \sim |z_{N-1} - z_N|^{4p+2} \left[a + \cdots \quad + \frac{b}{|z_{N-1} - z_N|^{4p}} \right]$$

Coordinate choice introduced by A-representation is not appropriate

X-representation

eigenvalues of X_1 $\approx$ 1d coordinates

$$\hat{X}_1 |X,\phi\rangle = X|X,\phi\rangle \;,\quad \hat{\Psi}|X,\phi\rangle = \phi|X,\phi\rangle$$

$$(X_2)_{ij} = -i\frac{\partial}{\partial (X_1)_{ij}}$$

Diagonalize X

$$\underset{\nearrow \text{unitary}}{X = U x U^{-1}} \qquad x_{ij} = x_i\delta_{ij}$$

$$\langle 2p+1|2p+1\rangle = \int [dX_{ij}]\, d\phi_\ell\, d\bar{\phi}_\ell \;\langle 2p+1|X,\phi\rangle\langle X,\phi|2p+1\rangle$$

$$= \int \underbrace{[dX_{ij}]}_{\substack{\shortparallel \\ dx_i \prod_{i<j}(x_i-x_j)^2 dU}} \prod_{i<j}(x_i-x_j)^{4p} e^{-Bx_i^2} \underbrace{d\bar{\phi}\, d\phi\, (\bar{\phi}_i\phi_i)^{2p} e^{-\bar{\phi}\phi}}_{x\text{-ind}}$$

$$\Rightarrow \Psi(x_1,\dots,x_N) = \underbrace{\prod_{i<j}(x_i-x_j)^{2p+1} e^{-\frac{B}{2}\sum_i x_i^2}}_{\text{ground state of Calogero model}}$$

$$\Psi(z_1,\dots,z_N) \sim e^{-\sum_i \frac{|z_i|^2}{2}} \left[e^{\frac{1}{4B}\sum_i \frac{\partial^2}{\partial x_i^2}} \prod_{i<j} (x_i - x_j)^{2p+1} \right]_{x_i = \frac{z_i}{\sqrt{2B}}}$$

- $\Psi(z_1 .. z_N)$ is holomorphic
- For $p=0$ $\quad \Psi \sim \prod_{i<j} (z_i - z_j) e^{-\sum_i \frac{|z_i|^2}{2}} = (\nu=1)$ Laughlin

For $p \neq 0$ $\quad \Psi \sim \left[(z_i-z_j)^{2p+1} + "(z_i-z_j)^{2p-1}" + \dots + "(z_i-z_j)" \right] e^{-\sum_i \frac{|z_i|^2}{2}}$

long distance agrees with $\nu = \frac{1}{2p+1}$ Laughlin

short distance different

What about excited states ?

In X-representation <u>Matrix model = Calogero model</u>

$$H = \frac{\omega}{2}\left[-\frac{1}{B}\frac{\partial^2}{\partial X_{ij}\,\partial X_{ji}} + B\,\mathrm{Tr}\,X^2\right]$$

$$\sum_{ij}\frac{\partial^2}{\partial X_{ij}\,\partial X_{ji}} = \frac{1}{\Delta}\sum_k \frac{\partial^2}{\partial x_k^2}\Delta - \sum_{k\neq l}\frac{J^{kl}J^{lk}}{(x_k-x_l)^2}$$

$$\left(\Delta = \prod_{k<l}(x_k-x_l)\right)$$

$$\left(X = U x U^{-1}\right)$$

$$\left[J^{kl}, U_{ij}\right] = U_{ik}\,\delta_{lj}$$

SU(N) singlets $<$ $S_n = \mathrm{Tr}\,X^n$; $\Xi = \epsilon^{i_1\ldots i_N}\,\bar\Phi_{i_1}\,(\bar\Phi X)_{i_2}\ldots(\bar\Phi X^{N-1})_{i_N}$

$$= \Delta \prod_{i=1}^{N}(\bar\Phi U)_i$$

Physical state $= f(S_n)\ \Xi^{2p}$

$$J^{kl}J^{lk}\ \Xi^{2p} = 2p(2p+1)\ \Xi^{2p}$$

H on physical states $=$ Calogero model

$$H = \frac{\omega}{B}\,\Delta^{-1}\Big[\underbrace{-\frac{1}{2}\sum\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\sum_{i\neq j}\frac{2p(2p+1)}{(x_i-x_j)^2} + \frac{1}{2}B^2\sum_i x_i^2}_{\text{Calogero model}}\Big]\Delta$$

Matrix model $=$ Calogero model

$\Vert$?

QHE

Matrix model

a) qualitative features of Laughlin theory

b) quantization of ν^{-1}

Does it provide microscopic description of Laughlin theory ?