FUZZY PHYSICS

WHAT IS IT : IDENTITY AND ORIGINS

FUZZY PHYSICS IS BASED ON COMPACT SPACETIME MANIFOLDS WHICH ARE QUANTISED.

THE EARLIEST SUGGESTION THAT SPACE-TIME ITSELF HAS QUANTUM FEATURES / UNCERTAINTY RELATIONS : IN A LETTER FROM HEISENBERG TO PAULI WHICH SUGGESTS

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}$$ (MODERN MOYAL PLANE).

SNYDER REVIVED THIS FORMULA IN A PHYS. REV. PAPER LATER.

IDEA : AT PLANCK SCALE, SPACETIME MAY HAVE A QUANTUM STRUCTURE CHARACTERISTIC OF QUANTUM PHYSICS.

CURRENT REVIVAL FROM

- NONCOMMUTATIVE GEOMETRY (CONNES) AND ITS APPLICATION TO 2-SPHERE S^2 (MADORE). S^2 IS SPATIAL SLICE OR EUCLIDEAN SPACETIME.

- D-BRANES OF STRING THEORY.

<u>THE FUZZY 2-SPHERE S_F^2 : THE DETAILS</u>

LET US BEGIN WITH $\mathbb{C}^2$, WITH COORDINATES (z_1, z_2).

WE CAN <u>QUANTISE</u> IT REPLACING $z_i, \bar{z}_i$ WITH OSCILLATORS $a_i, a_i^\dagger$:

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \hbar \delta_{ij}.$$

CONSIDER $\vec{x} = z^\dagger \vec{\tau} z$ FOR FIXED

$$R^2 = \sum |z_i|^2 = z^\dagger \cdot z.$$

IT DESCRIBES A SPHERE OF RADIUS R:

$$\vec{x} \cdot \vec{x} = R^2$$

FOR FUZZY SPHERE S_F^2

COORDINATE $\hat{\vec{x}} = a^+ \vec{\tau} a$, $\vec{\tau}$ = PAULI MATRICES.

NOW $\hat{\vec{x}} \cdot \hat{\vec{x}} = \hbar^2 N(N+2)$, $N = \frac{1}{\hbar} a^\dagger a$ = # OPERATOR.

SO RADIUS R IS GIVEN BY

$$R^2 = \hbar^2 N(N+2).$$

ROLE OF QUANTISATION: $\hat{x}_i$'S DON'T COMMUTE, SPHERE S_F^2 FUZZY:

$$\boxed{[\hat{x}_i, \hat{x}_j] = (2\hbar) i \varepsilon_{ijk} \hat{x}_k = 2\left[\frac{R}{\sqrt{N(N+2)}}\right] i \varepsilon_{ijk} \hat{x}_k}$$

BOX 1

FUZZY SPHERE S_F^2: ALGEBRA GENERATED BY $\hat{x}_i$'S.

$\boxed{\text{WE HAVE A SEQUENCE OF FINITE-DIMENSIONAL MATIX APPROXIMATIONS TO } S^2, \text{ INDEXED BY } n.}$

R

$\star$- PRODUCT ON S^2_F : FUZZY ALGEBRA = DEFORMED $C^\infty(S^2)$.

AN $(n+1) \times (n+1)$ MATRIX HAS NO PARTICULAR REFERENCE TO S^2.

COHERENT STATES AND $\star$ - PRODUCTS ARE GOOD WAYS TO HIGHLIGHT THOSE FEATURES, ASSOCIATED WITH S^2.

COHERENT STATES FOR QUANTISED / FUZZY $\mathbb{C}^2$, $\mathbb{C}^2_F$:

$$z = (z_1, z_2)$$

$$|z\rangle = e^{z \cdot a^+ - \bar{z} \cdot a} |0\rangle,$$

$$a_i |z\rangle = z_i |z\rangle, \quad \langle z | z \rangle = 1.$$

COHERENT STATE $\ast$ - PRODUCT INTRODUCED

USING THE MAP

$$\hat{A} \to A \quad , \quad A(z, \bar{z}) = \langle z | \hat{A} | z \rangle$$

OF OPERATORS TO FUNCTIONS, A, ON $\mathbb{C}^2$.

THEN $\ast$ DEFINED BY

$$\langle z | \hat{A} \hat{B} | z \rangle = (A \ast B)(z, \bar{z}).$$

A STANDARD CALCULATION GIVES

$$A \ast B(z, \bar{z}) = A \, e^{\frac{\hbar}{2} \overleftarrow{\partial}_z \cdot \overrightarrow{\partial}_{\bar{z}}} B(z, \bar{z}) \, ,$$

$$\xi \, \overleftarrow{\partial}_z \cdot \overrightarrow{\partial}_{\bar{z}} \, \eta \, (z, \bar{z}) := \frac{\partial \xi}{\partial z_1}(z, \bar{z}) \, \frac{\partial \eta}{\partial \bar{z}_1}(z, \bar{z}).$$

FOR S_F^2, OPERATORS COMMUTE WITH P_n, PROJECT~~ING~~OR TO $N = n$. SO CAN PROJECT $|z\rangle$ TO $N = n$ FOR S_F^2 - COHERENT STATE.

THE PROJECTED STATE $P_n|z\rangle$ ON NORMALISING BECOMES S_F^2 OR PERELOMOV COHERENT STATE

$$|z\rangle_n = \frac{(z\cdot a^+)^n}{\sqrt{n!}}|0\rangle, \quad \sum |z_i|^2 = 1.$$

IF $\hat{A} \in S_F^2$, i.e. A POLYNOMIAL IN (OPERATOR) $\hat{x}_i$, ITS FUNCTION A IS A FUNCTION ON S^2:

$$A(\vec{n}) = {}_n\langle z|\hat{A}|z\rangle_n, \quad \vec{n} = z^+\vec{\tau}z, \left.\begin{matrix} \vec{n}\cdot\vec{n} = 1, \\ = x_i/|\vec{x}| = \hat{x}_i/|\hat{\vec{x}}|. \end{matrix}\right\}$$

BECAUSE RHS INVARIANT UNDER $z \to z e^{i\theta}$.

$*$ - PRODUCT ON S_F^2 CAN BE WORKED OUT:

(PRESNAJDER; BALACHANDRAN ET AL.)

$$\boxed{\begin{aligned} (A * B)(\vec{n}) &= \sum_{m=0}^{n} \frac{(n-m)!}{m!\,n!}(\partial_{a_1}\cdots\partial_{a_m}A)K_{a_1b_1}\cdots K_{a_mb_m}(\partial_{b_1}\cdots\partial_{b_m}B)\,(\vec{n}) \\ \partial_a &= \frac{\partial}{\partial n_a}, \quad K_{ab} = \left(\frac{\theta\cdot n\,(\theta\cdot n \mp 1)}{2}\right)_{ab}, \quad \theta(\alpha) = \text{SPIN-1 ANGULAR MOMENTUM.} \end{aligned}}$$

$m=0$ TERM COMMUTATIVE PRODUCT $(AB)(\vec{n})$.

REST CORRECTIONS $O(\frac{1}{n})$.

FUZZY SPHERE : DEFORMED SPHERE ALGEBRA, DEFORMATION PARAMETER $1/n$.

OF WHAT USE IS FUZZY SPHERE ?

REGULARISATION OF QUANTUM FIELD THEORIES (QFT,

PLANCK AND BOSE REVEALED TO US: QUANTISATION INTRODUCES SHORT DISTANCE CUT-OFF: NO. OF STATES IN PHASE SPACE VOLUME $\Delta p \Delta q = \Delta p \Delta q / \hbar$.

SO QFT ON S_F^2 SHOULD BE A REGULARISED VERSION OF QFT ON S^2. S^2 CAN BE EUCLIDEAN SPACETIME OR SPATIAL SLICE.

THAT IS SO : S_F^2 FINITE-DIMENSIONAL.

PRELUDE TO EUCLIDEAN ACTIONS ON S_F^2

• ELEMENTS OF S_F^2 CORRESPOND TO FUNCTIONS ON S^2. THEY ARE WAVE FUNCTIONS OR SCALAR FIELDS.

• FOR EACH ELEMENT $m \in S_F^2$, THERE ARE 2 OPERATORS $m^{L,R}$ ON S_F^2:

$$m^L \alpha = m\alpha, \qquad m^R \alpha = \alpha m, \qquad \alpha \in S_F^2.$$

• IF $L_i \in S_F^2$ ARE ANGULAR MOMENTUM OPERATORS, ORBITAL ANGULAR MOMENTUM IS

$$\mathcal{L}_i = L_i^L - L_i^R, \qquad \mathcal{L}_i \alpha = [L_i, \alpha].$$

• ON S^2, A REAL SCALAR FIELD EUCLIDEAN ACTION IS

$$S = +\int d\Omega_{\hat{n}} \left[-(\mathcal{L}_i \varphi)^2(\vec{n}) + \mu^2 \varphi^2 + \lambda \varphi^4 \right] \qquad \Bigg| \qquad \mathcal{L}_i = -i(\vec{r} \wedge \vec{\nabla})_i$$

PARTITION FUNCTION

$$Z = \int d\varphi \; e^{-S}$$

THE FUZZY SCALAR FIELD

THE SCALAR FIELD Φ IS $(n+1) \times (n+1)$ (HERMITEAN) MATRIX. ABOVE ACTION TRANSLATES TO

$$S^{\#}_F = \frac{4\pi}{(n+1)} \left\{ \overset{(-)}{+}\, \mathrm{Tr}\left[(\mathcal{L}_i \Phi)^2 + \mu^2 \Phi^2 + \lambda \Phi^4 \right] \right\}$$

WITH COHERENT STATES, IT LOOKS LIKE

$$S^{*}_F = \int d\Omega_{\hat{n}} \left\{ -(\mathcal{L}_i \varphi) * (\mathcal{L}_i \varphi) + \mu^2 \varphi * \varphi + \lambda\, \varphi * \varphi * \varphi * \varphi \right\},$$

$$\varphi(z, \bar{z}) = {}_n\langle z | \Phi | z \rangle_n = \varphi(\vec{n}).$$

RENORMALISATION STUDIES ON THIS ACTION ARE DUE TO VAIDYA & YDRI; CHU, MADORE & STEINACKER; AND DOLAN ET AL.

<u>IMPORTANT RESULT</u>: SO-CALLED UV-IR MIXING CAN BE TRIVIALLY REMOVED.

NUMERICAL WORK CURRENTLY IN PROGRESS BY XAVIER MARTIN AND DENJOE O'CONNOR.

TECHNICAL REMARKS : ÖSTERWALDER - SCHRADER AXIOMS (OSA)

A EUCLIDEAN QUANTUM FIELD THEORY (EQFT) FULFILLING OSA HAS A REAL TIME VERSION.

PARTITION FUNCTION OF ACTION $\mathcal{S} = -\int d\Omega_{\hat{n}} (\mathcal{L}_i \varphi)^2$ ON S^2 FULFILLS OSA, AND HAS REAL TIME VERSION

$$\int dt \int_0^{2\pi} d\theta \left[(\partial_t \varphi)^2 - (\partial_\theta \varphi)^2 \right]^{*}. \qquad (*)$$

THE FUZZY VERSION $\mathcal{S}_F$ OF $\mathcal{S}$ ALSO FULFILLS OSA AND GIVES A REGULARISED EUCLIDEAN VERSION OF $(*)$.

<u>NOTE</u> IF $e^{i\varphi(t,\theta)} \in S^1_{*}$ IS THE ACTUAL FIELD $[\varphi(t,\varphi) \approx \varphi(t,\theta) + 2\pi]$, $*$ CONFORMAL QFT

THE DIRAC OPERATOR: GINSPARG-WILSON (GW) SYSTEM

ON S^2, IN CONTINUUM, DIRAC AND CHIRALITY OPERATORS $\mathcal{D}$ AND γ ARE

$$\mathcal{D} = (\sigma \cdot \mathcal{L} + 1), \quad \gamma = \sigma \cdot n, \quad \boxed{\gamma\mathcal{D} + \mathcal{D}\gamma = 0}$$

BOX 2

γ IS IDEMPOTENT: $\gamma^2 = 1$.

$\mathcal{D}$ ACTS ON 2-COMPONENT SPINORS $\psi \in C^\infty(S^2) \otimes \mathbb{C}^2$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_i \in C^\infty(S^2).$$

THERE IS AN INDEX THEOREM. BY BOX 2,

$$\mathcal{D}\psi = E\psi \Leftrightarrow \mathcal{D}(\gamma\psi) = -E(\gamma\psi)$$

$$(\gamma\psi = \psi \Leftrightarrow \gamma(\mathcal{D}\psi) = -(\mathcal{D}\psi).)$$

$$\Rightarrow [\gamma, |\mathcal{D}|] = 0 \qquad \text{SO}$$

$$\underset{|\mathcal{D}|=|E|\neq 0}{\mathrm{Tr}\ \gamma} = \underset{|\mathcal{D}|=|E|\neq 0}{\mathrm{Tr}\ \mathcal{D}\gamma\mathcal{D}^{-1}} = -\underset{|\mathcal{D}|=|E|\neq 0}{\mathrm{Tr}\ \gamma} = 0$$

So $\quad \mathrm{Tr}\,\gamma = \underset{|\mathcal{D}|=0}{\mathrm{Tr}}\,\gamma = \underset{\mathcal{D}=0}{\mathrm{Tr}}\left(\frac{1+\gamma}{2} - \frac{1-\gamma}{2}\right)$

$= \# n_L$ OF LEFT $-$ $\# n_R$ OF RIGHT CHIRAL ZERO MODES.

$$\boxed{\text{USES ONLY} \Big\} \; ① \; \gamma\mathcal{D} + \mathcal{D}\gamma = 0 \; ; \; ② \; \gamma^2 = 1 \text{ ON } \mathcal{D}=0 \text{ SUBSPACE}}$$

$\downarrow$

① & ② VALID WITH GAUGE FIELDS, INSTANTONS, ...

<u>FUZZY DIRAC</u> (Bal, Immirzi, TRG, Ydri)

$D = \sigma\cdot\mathcal{L} + 1 = \sigma\cdot(L^L - L^R) + 1$

ACTS ON $\quad Mat(n+1) \otimes \mathbb{C}^2 \ni \Psi = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}, \; \Psi_1 \in Mat(n+1).$

NOW

$$D = \left[\frac{n+1}{2}\right]\left\{\gamma^L + \gamma^R\right\},$$

$$\gamma^L = \frac{2}{n+1}\left(\sigma\cdot L^L + \tfrac{1}{2}\right) \; ; \quad \gamma^R = \frac{2}{n+1}\left(-\sigma\cdot L^R + \tfrac{1}{2}\right).$$

$$(\gamma^{L,R})^2 = 1.$$

AS $\quad n \to \infty, \quad \gamma^L = -\gamma^R = \vec{\sigma}\cdot\vec{n}$

THE GW SYSTEM IS JUST ALGEBRA OF 2 IDEMPOTENTS, CALL THEM IN GENER $\gamma^{L,R}$. THEN DIRAC OPERATOR IS

$$\mathcal{D} = \frac{1}{a}(\gamma^L + \gamma^R), \quad a: \text{LATTICE SPACING} = \left(\frac{2}{n+1}\right)^{\pm 1}$$

HE CHIRALITY

$$\Gamma = \frac{1}{2}(\gamma^L - \gamma^R), \Rightarrow [\Gamma, \mathcal{D}]_+ = \Gamma\mathcal{D} + \mathcal{D}\Gamma = 0.$$

ALSO

$$\boxed{\Gamma^2 = 1 \text{ IF } \mathcal{D} = 0 \text{ OR } \gamma^L = -\gamma^R \text{ ON A SUBSPACE.}}$$

SO INDEX THEOREM:

$\boxed{\mathrm{Tr}\ \Gamma = n_L - n_R}$ WORKS WITH FUZZY GAUGE FIELDS, INSTANTONS.

INTEGRATED $\overset{U(1)}{\vee}$ CHIRAL ANOMALIES IN GAUGE THEORIES $= n_L - n_R$.

ILLUSTRATION : U(1) ANOMALY WITH INSTANTONS

BAL & VAIDYA & YDRI

FOR DIRAC SPINORS FOR INSTANTON NUMBER $N > 0$, WE TAKE N SETS OF COMMUTING PAULI MATRICES $\vec{\tau}^{(i)}$, $i = 1, 2, \cdots, N$.

THEN CONSIDER

$$\psi = \chi \otimes \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{\text{FOR } \vec{\tau}^{(i)}} \otimes \underset{\uparrow \text{ FOR SPIN}}{\mathbb{C}^2}$$

$$\chi \in Mat\,(n+1).$$

IF $P\left(\frac{n+N}{2}\right)$ = PROJECTOR COUPLING $\vec{L}^L_i$ AND $\vec{\tau}^{(i)}/2$'S TO MAXIMUM "ANGULAR MOMENTUM $(n+N)/2$, SPINORS FOR INSTANTON # N:

$$P\left(\frac{n+N}{2}\right)\psi.$$

AND DIRAC

$$D = \sigma\cdot\left(\vec{L}^L + \sum_i \frac{\vec{\tau}^{(i)}}{2} - \vec{L}^R\right) + 1.$$

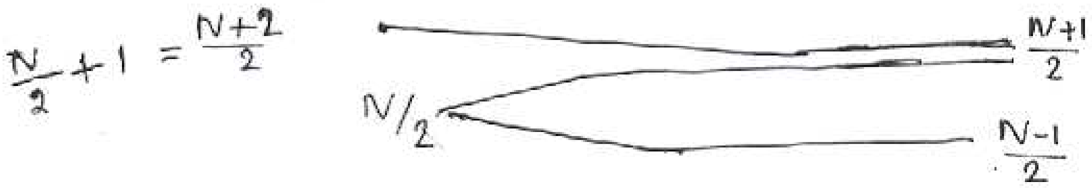
$$\boxed{\begin{array}{c}\text{NEW CONSERVED ANGULAR MOMENTUM} \\ \vec{J} = \vec{L}^L + \sum \vec{\tau}^{(i)}/2 - \vec{L}^R + \vec{\sigma}/2.\end{array}}$$

IN CONTINUUM, ADDITION OF $\vec{\tau}^{(i)}$'S IN $\vec{J}$ IS CALLED "<u>MIXING OF SPIN AND ISOSPIN</u>".

CAN SHOW THERE IS A ROTATIONALLY INVARIANT Γ ~~CO~~ ANTICOMMUTING WITH D, AND NEEDED PROPERTIES. ~~SO~~

<u>J- VALUES</u>

$\vec{L}^L$ = LEFT	$\vec{L}^R$	TOTAL LEFT+RIGHT	$\vec{J}$
$\frac{n+N}{2}$	$\frac{n}{2}$	$n + N/2$	



$J = \frac{N-1}{2}$ FOR FIXED J_3 HAS NO DEGENERACY. IF $D\Psi_{J,J_3} = E\Psi_{J,J_3}$,

$D(\Gamma\Psi_{J,J_3}) = -E(\Gamma\Psi_{J,J_3})$. BUT $\Gamma\Psi_{\frac{N-1}{2},J_3} = \pm\Psi_{\frac{N-1}{2},J_3}$

$\Rightarrow E = 0$; $\boxed{\text{N ZERO MODES}}$ CAN SHOW LEFT-CHIRAL.

$n_L - n_R$ = INSTANTON # N. (INDEX THM).

FOR INSTANTON # $-N$, USE PROJECTOR $P\left(\frac{-N+n}{2}\right)$, GET N RIGHT-CHIRAL ZERO MODES.

FUZZY TOPOLOGY

QUANTUM PHYSICS ALLOWS POSSIBILITY THAT $\exists$ STATES LOCALISED NEITHER IN ONE NOR ANOTHER TOPOLOGY.

EVEN DIMENSION CAN FLUCTUATE.

EXAMPLES USING OPERATOR ALGEBRAS WERE PRODUCED BY BALACHANDRAN, BIMONTE, MARMO, SIMONI, IN MID-80'S. AT AROUND SAME TIME, MADORE & CO. DISCUSSED FLUCTUATIONS BETWEEN FUZZY S^2 & TORUS T^2.

EXAMPLE S^2_F

$$|n_1, n_2\rangle = \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}}{(n_1!\, n_2!)^{1/2}} |0\rangle \qquad n_1 + n_2 = n$$

WAVE FUNCTIONS : SPAN OF $|n_1, n_2\rangle\langle n_1', n_2'|$.

WAVE FUNCTIONS $(n+1) \times (n+1)$ MATRICES.

S^2 IS $\mathbb{C}P^1$.

$\underline{\mathbb{C}P^2}$ USE 3 OSCILLATORS b_i, $i = 1, 2, 3$.

WAVE FUNCTIONS OPERATORS ON

$$\prod \frac{(b_i^+)^{m_i}}{\sqrt{m_i!}} |0\rangle \quad , \quad \sum m_i = m$$

$$= |m_1, m_2, m_3\rangle$$

A WAVE FUNCTION $|m_1, m_2, m_3\rangle\langle m_1', m_2', m_3'|$.

THEY ARE MATRICES OF SIZE $\frac{(m+1)(m+2)}{2} \times$
$(m+1)(m+2)/_2$.

BY CHOOSING n & m, WE CAN MAKE SAME SIZE MATRIX SERVE AS S^2_F OR $\mathbb{C}P^2_F$ WAVE FUNCTION. THE INTERPRETATION DEPENDS ON OBSERVABLES WE LOOK AT.

A $\mathbb{C}P^1_F \simeq S^2_F$ STATE VECTOR APPROXIMATELY LOCALISED AT $z^+ \vec{\tau} z$ IS

$$|z\rangle_{n\,n}\langle z| \qquad (1)$$

FOR $\mathbb{C}P^2$, A SIMILAR WAVE FUNCTION IS

$$|\eta\rangle_{m\,m}\langle\eta| \;,\quad \eta = (\eta_1, \eta_2, \eta_3)\;,\quad \sum |\eta_i|^2 = 1. \qquad (2)$$

$$|\eta\rangle_m = \cancel{\Pi} \frac{[\eta_i (b_i^+)]^m}{[m!]^{1/2}} |0\rangle . \qquad (3)$$

CHOOSE n, m SO THAT (1), (2) SAME SIZE MATRIX. THEN PROBABILITY OF FINDING η OF $\mathbb{C}P^2$ IN STATE LOCALISED AT $z^+ \vec{\tau} z$ OF $\mathbb{C}P^1$ IS

$$\left| {}_n\langle z|\eta\rangle_m \right|^4 \quad !$$

NOTE: $\mathbb{C}P^{1,2}$ HAVE DIFFERENT DIMENSIONS!

<u>LARGE</u> <u>SIZE</u> <u>MATRICES</u> <u>SUPPORT</u> <u>A PROLIFE-RATING # OF FUZZY MANIFOLDS</u>: <u>$\mathbb{C}P^N$</u>, <u>$SU(N)/U(1)\times\cdots\times U(1)$</u> ...

$\boxed{\text{WE CAN POSE AND ANSWER APPARENTLY EXOTIC QUESTIONS IN FUZZY FRAMEWORK.}}$

REMARKS

- $|\zeta\rangle_n$, $|\eta\rangle_m$ A PRIORI LIVE IN DIFFERENT VECTOR SPACES. SO WE NEED A MAP T FROM ONE TO THE OTHER.

  A CHOICE MAY BE: IF $|\alpha\rangle_m$, $|i\rangle_n$ LAPLACIAN EIGENSTATES FOR EIGENVALUES $E_\alpha^{(m)}$, $E_i^{(n)}$,

$$T\,|\alpha\rangle_m = \sum_n \exp\left[-\beta\,|E_\alpha^{(m)} - E_i^{(n)}|\right]|i\rangle_n,$$

  β A PARAMETER. ${}_n\langle\zeta|\eta\rangle_m \equiv {}_n\langle\zeta|T|\eta\rangle_m$.

- THERE ARE ALSO EXAMPLES OF ONE FUZZY MANIFOLD (S_F^4) FLUCTUATING INTO ANOTHER ($\mathbb{C}P_F^3$). DIMENSION FLUCTUATES.

WINDING UP

- MODELS OF SPACETIME BASED ON NONCOMMUTATIVE GEOMETRY IS SUGGESTED BY QUANTUM PHYSICS ITSELF, STRING PHYSICS AND ATTEMPTS TO REGULARISE QUANTUM FIELD THEORIES.

- THEY LEAD TO STRIKINGLY NOVEL SPACETIME MODELS.

- BUT LACKING CONTACT WITH EXPERIMENTS THEY REMAIN METAPHYSICAL THEORIES, JUST AS QUANTUM GRAVITY AND STRING PHYSICS. SO THEIR MAIN USE MAY BE IN REGULARISING QFT'S.