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I.M.Sc. Workshop

LOOP EQUATIONS IN NON-COMMUTATIVE GAUGE THEORIES

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Why non-commutativity?

→ A consistent theory of quantum gravity is probably space-time noncommutative at a fundamental level.

→ String theory (which is an example of a perturbatively consistent quantum theory of gravity) has a fundamental scale and naturally gives rise to situations with space-time noncommutativity

→ Many nice features naturally present

Why Loop Eqns.?

→ In ordinary, commutative gauge theories, Wilson loops contain all of gauge-invariant content. The dynamics of Wilson loops is described by Loop Equations.

However, local gauge-invariant operators exist .....

→ In the non-commutative case, gauge-invariant operators are ~~essentially~~ non-local (at least for the Heisenberg type of non-comm. that we will discuss) in an essential way.

③

Thus in this case, Loop Equations are essential to discuss the dynamics of gauge-invt. operators.

→ Not a straight-forward generalization of ordinary commutative case because Wilson loops are both "open" & closed!

Plan

- Heisenberg non-commutativity and gauge theories
- Connection with matrix models
- Gauge-invariant observables
- Loop equations
- Large-N limit and correlators of "open" Wilson lines

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Heisenberg non-commutativity

Euclidean 4-dims.

$$[\hat{x}^\mu, \hat{x}^\nu] = i\,\theta^{\mu\nu}$$

$\hookrightarrow$ dims. of area

Canonical form : $\theta^{\mu\nu} \sim \left(\begin{array}{cc|cc} 0 & \theta_1 & & \\ -\theta_1 & 0 & & \\ \hline & & 0 & \theta_2 \\ & & -\theta_2 & 0 \end{array}\right)$

$$\Rightarrow \quad \left.\begin{array}{l} [\hat{x}^1, \hat{x}^2] = i\theta_1 \\ [\hat{x}^3, \hat{x}^4] = i\theta_2 \end{array}\right\} \text{ others commute}$$

$$a_1 \equiv \frac{\hat{x}^1 + i\hat{x}^2}{\sqrt{2\theta_1}} \quad , \quad a_2 \equiv \frac{\hat{x}^3 + i\hat{x}^4}{\sqrt{2\theta_2}}$$

$$[a_1, a_1^\dagger] = 1 = [a_2, a_2^\dagger]$$

$\rightarrow$ Hilbert space $\mathcal{H}$ of two decoupled harmonic oscillators

$\rightarrow$ fns. on $\{\hat{x}^\mu\}$ are operators on $\mathcal{H}$.

⑥

→ A derivative satisfying the usual properties can be defined

$$\hat{\partial}_\mu \hat{\phi}(\hat{x}) = -i\left(\theta^{-1}\right)_{\mu\nu} \left[\hat{x}^\nu, \hat{\phi}(\hat{x})\right]$$

→ Integration is trace over $\mathcal{H}$

→ Weyl ordering (⇒) description in terms of an auxilliary commutative space; but with a complicated product

$$\hat{\phi}(\hat{x}) \sim \int d^4x \; \phi(x) \; \delta^{(4)}(\hat{x}-x)$$

$$\int \frac{d^4k}{(2\pi)^4} \, e^{-ik.x} \, e^{ik.\hat{x}}$$

$$\mathrm{tr}\left(\delta^{(4)}(\hat{x}-x)\right) = 1$$

$$\mathrm{tr}\left(\delta^{(4)}(\hat{x}-x)\;\delta^{(4)}(\hat{x}-y)\right) = \delta^{(4)}(x-y)$$

$$\hat{\phi}_1(\hat{x})\,\hat{\phi}_2(\hat{x}) \sim \int d^4x \; \underbrace{\phi_1 * \phi_2(x)}_{\exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_{1\mu}\partial_{2\nu}\right)\phi_1(x_1)\phi_2(x_2)\big|_{x_1=x_2=x}} \; \delta^{(4)}(\hat{x}-x)$$

⑦

Non-Commutative Gauge Theories

- Gauge field $\hat{A}_\mu(\hat{x})$ is an operator on $\mathcal{H}$

- Gauge transformations are operator valued : $\hat{\Omega}(\hat{x})$, $\hat{\Omega}(\hat{x})\hat{\Omega}(\hat{x})^\dagger = 1$ for unitary groups

- $U(1)$ theory:

  $\hat{A}_\mu(\hat{x})$ and $\hat{\Omega}(\hat{x})$ do not commute

  $$\Rightarrow \left(\hat{\partial}_\mu \hat{A}_\nu(\hat{x}) - \hat{\partial}_\nu \hat{A}_\mu(\hat{x}) \right)$$

  is <u>not</u> gauge-invariant

⑧

- Gauge covariant quantity is

$$\hat{F}_{\mu\nu}(\hat{x}) \equiv \hat{\partial}_\mu \hat{A}_\nu(\hat{x}) - \hat{\partial}_\nu \hat{A}_\mu(\hat{x}) + i\underbrace{[\hat{A}_\mu(\hat{x}), \hat{A}_\nu(\hat{x})]}$$

This is non-zero like in nonabelian gauge theories

$$\underbrace{\hat{F}_{\mu\nu}(\hat{x}) \to \hat{\Omega}(\hat{x})\, \hat{F}_{\mu\nu}(\hat{x})\, \hat{\Omega}(\hat{x})^\dagger}$$

this is like nonabelian gauge theories

- In particular, take

$$\hat{\Omega}(\hat{x}) = e^{ik\cdot\hat{x}}$$

Then, $\hat{A}_\mu(\hat{x}) \to \hat{A}_\mu(\hat{x}+\theta k) - k_\mu$

$\Rightarrow$ Apart from a trivial shift, gauge transformation is equivalent to a translation. $\Rightarrow$ Absence of <u>local</u> gauge-invt. observables!

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Connection with Matrix Models

$\{X^\mu\} \rightarrow$ $N \times N$ hermitian matrices

$S \sim \mathrm{tr}[X^\mu, X^\nu]^2$

Classical eqn. : $[X^\mu, [X^\mu, X^\nu]] = 0$

Soln. : $[X^\mu, X^\nu] = \underbrace{i\theta^{\mu\nu} \mathbb{1}_{N\times N}}$

makes sense only in the Large-N limit

Fluctuations : $X^\mu = \hat{x}^\mu - \theta^{\mu\nu} \hat{A}_\nu(\hat{x})$

classical soln.

$\Rightarrow [X^\mu, X^\nu] = i\theta^{\mu\nu} - i\theta^{\mu\lambda}\theta^{\nu\sigma}\hat{F}_{\lambda\sigma}(\hat{x})$

$\Rightarrow S \sim \mathrm{tr}\left(\hat{F}_{\mu\nu}(\hat{x}) - \theta_{\mu\nu}\right)^2$

Gauge Invariance

$$X^\mu \to \mathcal{U} X^\mu \mathcal{U}^\dagger$$

← global $U(N)$ symm. of matrix action

$$\Downarrow$$

$$\mathcal{U}\left(\hat{x}^\mu - \theta^{\mu\nu} \hat{A}_\nu(\hat{x})\right)\mathcal{U}^\dagger$$

$$= \mathcal{U}\hat{x}^\mu \mathcal{U}^\dagger - \theta^{\mu\nu}\, \mathcal{U}\hat{A}_\nu(\hat{x})\,\mathcal{U}^\dagger$$

$$= \hat{x}^\mu - [\hat{x}^\mu, \mathcal{U}]\,\mathcal{U}^\dagger - \theta^{\mu\nu}\, \mathcal{U}\hat{A}_\nu(\hat{x})\,\mathcal{U}^\dagger$$

$$= \hat{x}^\mu - \theta^{\mu\nu}\underbrace{\left(\mathcal{U}\hat{A}_\nu(\hat{x})\,\mathcal{U}^\dagger + i\,\hat{\partial}_\nu \mathcal{U}\,\mathcal{U}^\dagger\right)}$$

gauge transformed potential!

⇒ Global $U(N)$ symm. of matrix model is inherited as <u>local</u> gauge symm. in the non-commutative theory!

Gauge-Invariant Observables

For finite N, the $U(N)$ invariant quantities are, e.g.

$$\mathrm{tr}\, X^\mu, \quad \mathrm{tr}(X^\mu X^\nu), \quad \ldots, \quad \mathrm{tr}(X^{\mu_1})^{n_1}(X^{\mu_2})^{n_2}\cdots$$

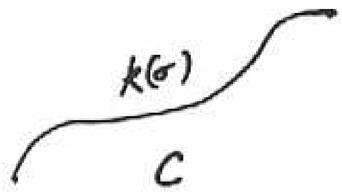
A more convenient quantity from which all of the above can be obtained by suitable manipulations is

$$\mathrm{tr}\left(e^{ik_1\cdot X}\, e^{ik_2\cdot X} \cdots\cdots e^{ik_n\cdot X}\right)$$

In fact, one can do a <u>smoother</u> job : $\{k_{1\mu}, k_{2\mu}, \ldots, k_{n\mu}\} \to \{k_\mu(\sigma)\}$

Then, the generic $U(N)$ invariant quantity is

$$\mathrm{tr}\left(P\, e^{i\oint_C d\sigma\, k_\mu(\sigma)\, X^\mu}\right)$$

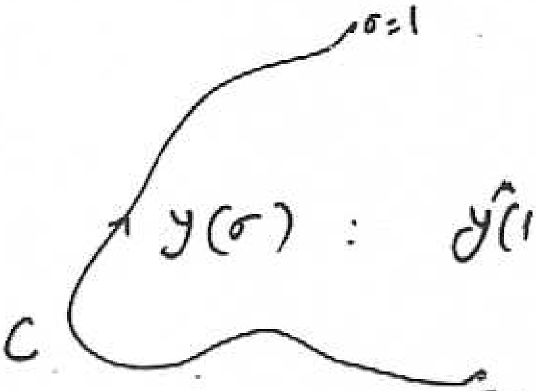


Substitute the form (for $N \to \infty$)

$$X^\mu = \hat{x}^\mu - \theta^{\mu\nu}\hat{A}_\nu(\hat{x})$$

This gives

$$\mathrm{tr}_{x}\left(P \exp\left\{ i \oint_C d\sigma\, \dot{y}^\mu(\sigma)\, \hat{A}_\mu\big(\hat{x}+y(\sigma)\big)\right\} e^{ik\cdot\hat{x}}\right)$$



$y(\sigma)$: $\hat{y}(1) - \hat{y}(0) = \theta^{\mu\nu}k_\nu$

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$\Rightarrow$ Generic gauge-invariant observable in NCGT is an <u>Open</u> Wilson line. (IIKT; Rey & Unge; Das & Rey; Dhar & Wadia; Andreev & Dorn)

<u>Gauge-invariance</u>

$$P\exp\{\cdots\}e^{ik\cdot\hat{x}} \rightarrow \hat{\Omega}(\hat{x}+y(0))\,P\exp\{\cdots\}\,\hat{\Omega}^{\dagger}(\hat{x}+y(1))\,e^{ik\cdot\hat{x}}$$

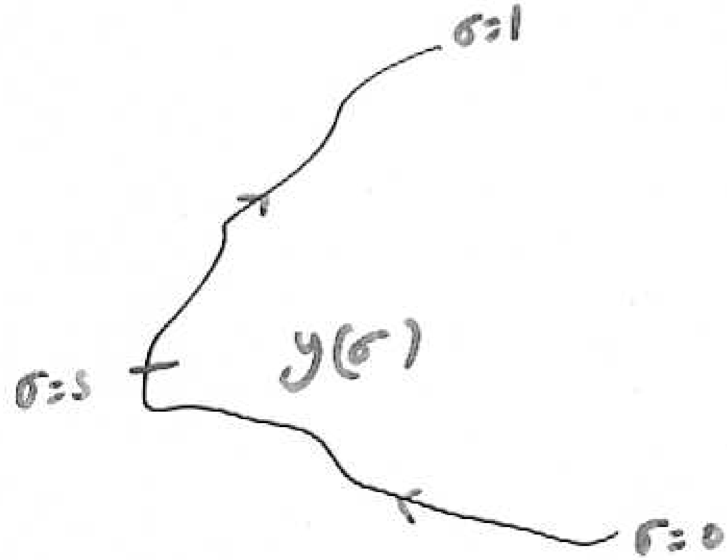
$$= \hat{\Omega}(\hat{x}+y(0))\,P\exp\{\cdots\}\,e^{ik\cdot\hat{x}}\,\hat{\Omega}^{\dagger}(\hat{x}+y(0))$$

$$\because\ \hat{\Omega}^{\dagger}(\hat{x}+y(1))\,e^{ik\cdot\hat{x}} = e^{ik\cdot\hat{x}}\,\hat{\Omega}^{\dagger}(\hat{x}+y(0))$$

using $y(1)-y(0)=\theta k$

cyclic symmetry

(c.f. $\mathrm{tr}\, AB = \mathrm{tr}\, BA$)

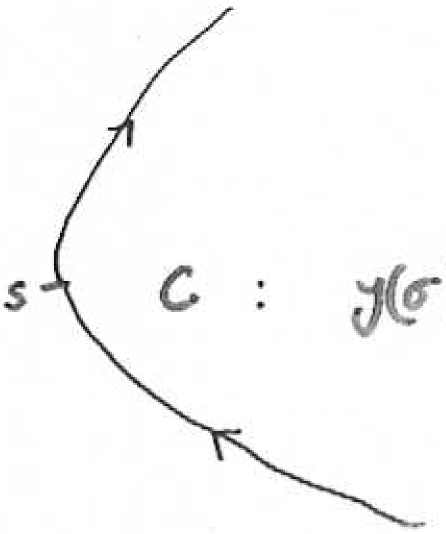


$$\mathrm{tr}\left\{ P\exp i\int_0^s d\sigma \cdots \quad \exp i\int_s^1 d\sigma \cdots \quad e^{ik\cdot\hat{x}} \right\}$$

$$= \mathrm{tr}\left\{ \underbrace{P\exp i\int_0^s d\sigma \cdots \; e^{ik\cdot\hat{x}}} \; \underbrace{P\exp i\int_s^1 d\sigma\, \dot{y}^\mu(\sigma)\hat{A}_\mu(\hat{x}+y(\sigma)-\theta k)} \right\}$$

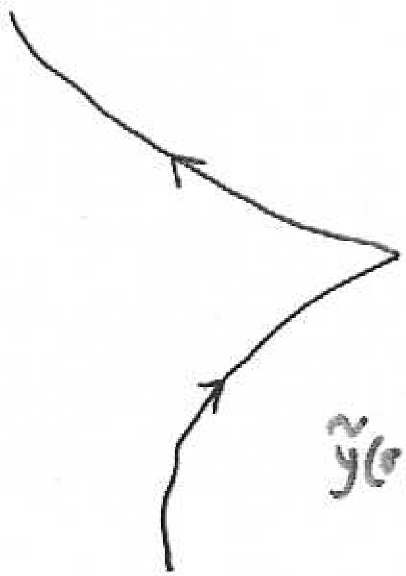
$$= \mathrm{tr}\left\{ \underbrace{P\exp i\int_s^1 d\sigma\, \dot{y}^\mu(\sigma)\hat{A}_\mu(\hat{x}+y(\sigma)-\theta k)} \; \underbrace{P\exp i\int_0^s d\sigma \cdots \; e^{ik\cdot\hat{x}}} \right\}$$

14a



$C \; : \; y(\sigma)$

$\downarrow$



$\tilde{C} \; : \; \tilde{y}(\sigma)$

$$\tilde{y}(\sigma) = y(s+\sigma) - y(1) + y(0), \quad 0 \le \sigma \le 1-s$$
$$= y(\sigma - 1 + s), \quad 1-s \le \sigma \le 1$$

Wilson line on C

$=$ Wilson line on $\tilde{C}$

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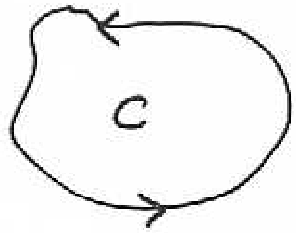
Loop Equations — Commutative case

Loop eqns. capture gauge-invt. non-pert. dynamics of gauge theories.

In commutative gauge theories, a generic gauge-invt. observable is a closed Wilson loop. $\mathrm{tr}\, P\, e^{i \oint_C dx^\mu A_\mu(x)}$



The loop eqn. relates a geometric deformation of the loop in spacetime to splitting & joining of loops.

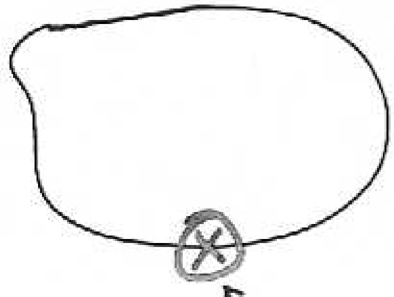
(15a)

Derivation of loop eqn. needs an action :

$$S \sim -\frac{1}{4} \operatorname{tr} \hat{F}^{2}_{\mu\nu}(\hat{x}) + \ldots$$

matter couplings, susy, etc.

We will use diagrammatic notation to discuss the eqn.



comes from the $\operatorname{tr}(\hat{F}^2_{\mu\nu}(\hat{x}))$ part of the action

$\hat{D}_\mu \hat{F}_{\mu\nu}(\hat{x})$

In any correlation fn. the insertion of this operator results in <u>splitting & joining</u> of loops.

Single loop

(i) No self-intersections

~ 0

(ii) Self-intersecting loop

$\sim \frac{1}{N^2}$

Splitting

Multiple loops

$\sim \frac{1}{N^2}$

Joining

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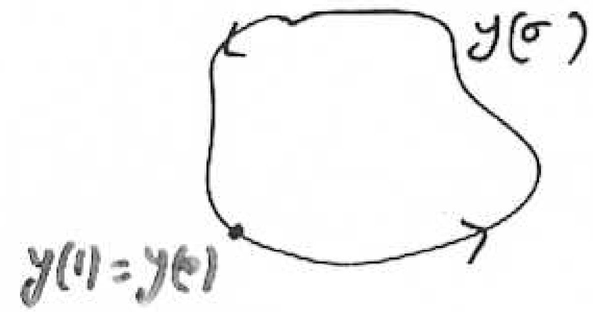
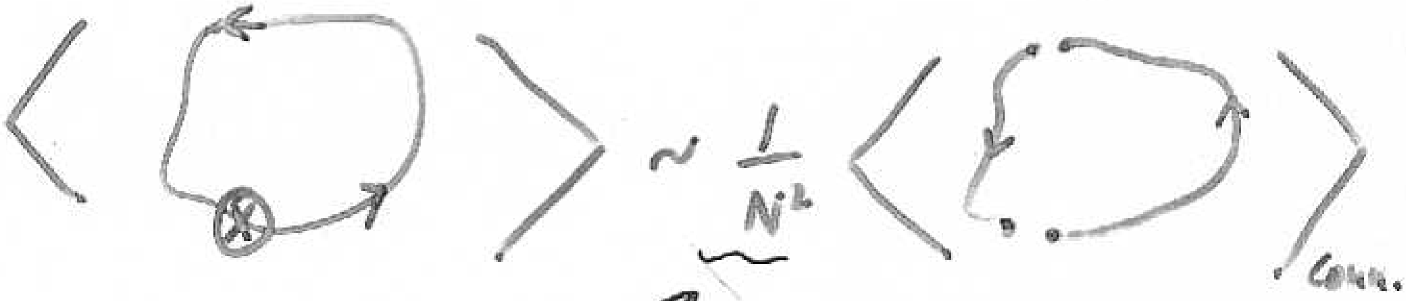
Non-Commutative Case

(i) closed loop



$\Rightarrow k_\mu = 0$



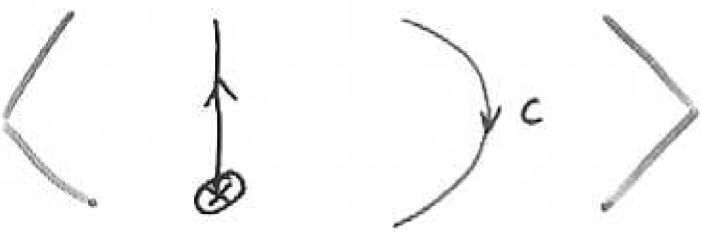
$\frac{1}{N^2\theta^2}$

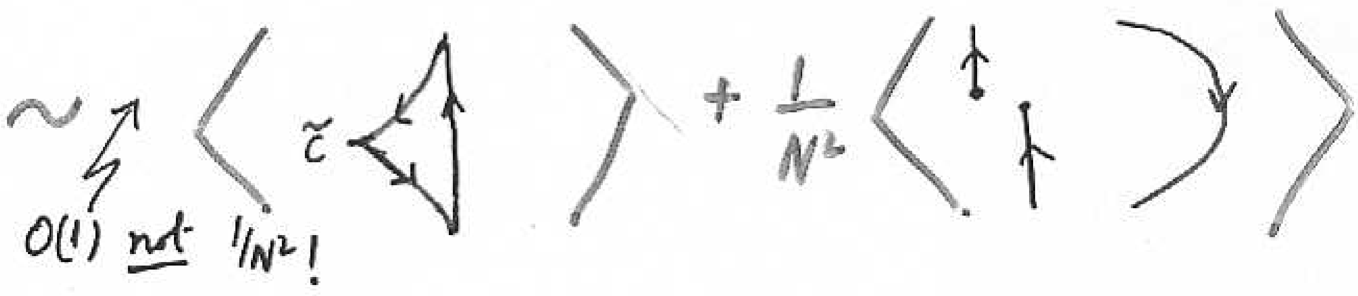
Two-point fn. of open Wilson lines!

For fixed N, at sufficiently small θ the large-N expansion breaks down (in general) and commutative limit is not recovered! UV-IR mixing

(ii) Open Wilson Lines

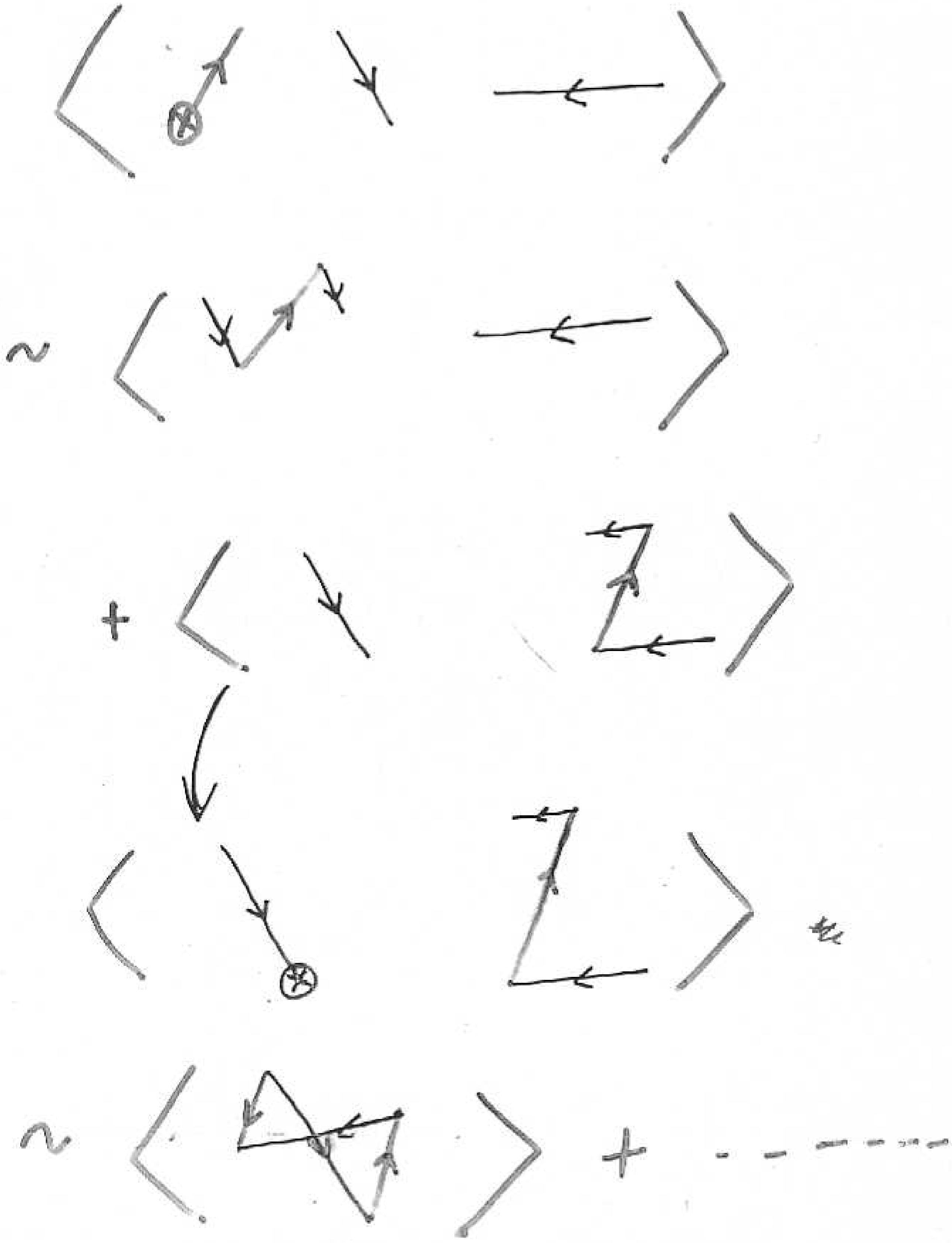
Expectation value of a single open Wilson line vanishes (momentum conservation).





$\Rightarrow$ In the $N \to \infty$ limit the 2-pt. fn. is related to the expectation value of a closed Wilson loop, the closed contour being made up by joining the open contours in a braided manner.

Higher pt. fns. (N → ∞ limit)

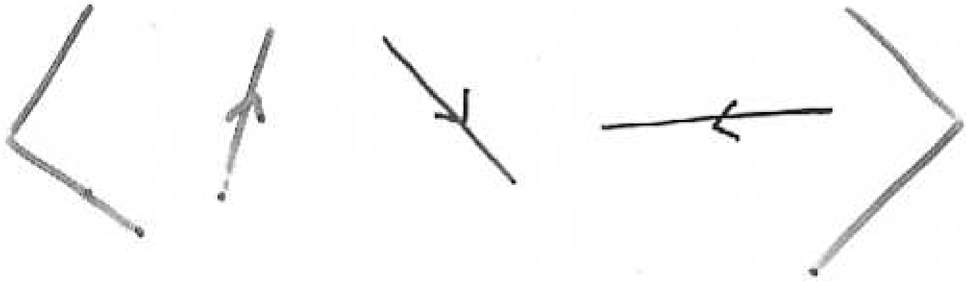


Thus, in the large-N limit, n-pt. correlators of Wilson lines can eventually be related to the expectation value of a closed Wilson loop (obtained from the Wilson lines by suitably joining them).

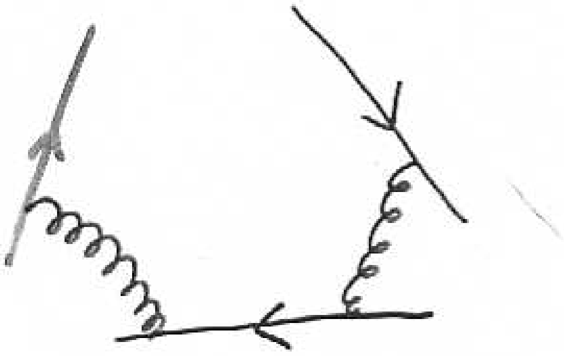
Perturbatively verified to the 2nd order in 't Hooft parameter. Proof to all orders needs to be worked out - - - -

Application

Consider the 3-pt. fn.

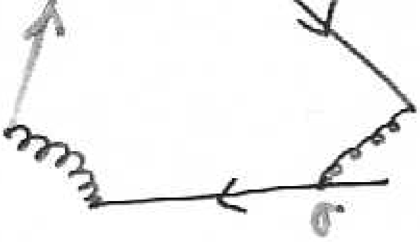


Lowest order diagrams

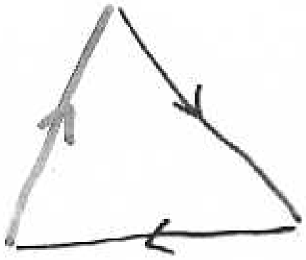
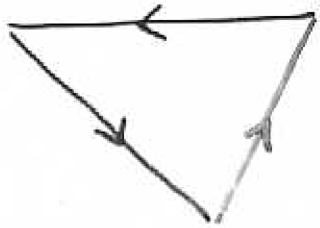


+ perm.

$\Downarrow$ By cyclic symm.



integration over σ in the amplitude

\#  + \# 

A more general statement follows from Loop eqns., which is valid non-perturbatively in 't Hooft parameter, but at large N.

Summary

- Loop eqns. in NCGT involve both closed and open Wilson lines
- At large-N, correlation of open Wilson lines can be expressed entirely in terms of expectation value of a closed loop.
- Geometrical meaning of correlation of open Wilson lines?
- AdS/CFT in the presence of a B-field ....