GAUGE SYMMETRIES IN NONCOMMUTATIVE GAUGE THEORIES

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GAUGE SYMMETRY (COMM. SPACE)

- $S = -\frac{1}{4}\int \mathrm{Tr} F^{\mu\nu} F_{\mu\nu} + \int \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$

  $D_\mu\psi = \partial_\mu\psi + igA_\mu\psi$; $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$

- Lie algebra valued G.T.

  $\alpha(x) = \alpha^a(x)T^a$, $[T^a, T^b] = if^{abc}T^c$

- G.T. of Fields

  $\delta\psi(x) = -ig\alpha(x)\psi(x)$; $\delta A_\mu = D_\mu\alpha = \partial_\mu\alpha + ig[A_\mu, \alpha]$

- Use comultiplication (Leibniz) rule

  $\delta(AB) = (\delta A)B + A(\delta B)$

  $\Rightarrow \delta S = 0$

  (commlt. rule defines the corresponding Hopf alg.)

NC GAUGE SYMMETRY

- $S = -\frac{1}{4}\int \mathrm{Tr}(F_{\mu\nu}\star F^{\mu\nu}) + \int \bar{\psi}\star(i\gamma^{\mu}D_{\mu} - m)\star\psi$
  $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + i(A_{\mu}\star A_{\nu} - A_{\nu}\star A_{\mu})$

- STAR PRODUCT
  $A\star B = \mu\{e^{\frac{i}{2}\theta^{\rho\sigma}\partial_{\rho}\otimes\partial_{\sigma}} A\otimes B\}$
  $\mu(A\otimes B) = A.B$; $\theta^{\rho\sigma} = -\theta^{\sigma\rho}$ is 'x' indep.

- FIRST OPTION
  Deformed G.T.
  $\delta\psi = -i\alpha\star\psi$; $\delta A_{\mu} = \partial_{\mu}\alpha + i(A_{\mu}\star\alpha - \alpha\star A_{\mu})$
  But Leibniz rule is undeformed;
  $\delta(A\star B) = (\delta A)\star B + A\star(\delta B)$
  ($\star$- product does not transform under the deformed G.T.)
  ($\delta_{\star} = 0$).

Finite G.T. on an adjoint field ϕ

$\phi(x) \rightarrow u(x) \star \phi(x) \star u^{-1}(x)$
with $u(x) \star u^{-1}(x) = u^{-1}(x) \star u(x) = 1$

- Ensures undeformed Leibniz rule:

$$\begin{aligned}\phi'_1 \star \phi'_2 &= [u \star \phi_1 \star u^{-1}] \star [u \star \phi_2 \star u^{-1}] \\ &= u \star (\phi_1 \star \phi_2) \star u^{-1} \\ &= (\phi_1 \star \phi_2)'\end{aligned}$$

Take its inf. version:

$\phi'_1 = \phi_1 + \delta\phi_1,\ \phi'_2 = \phi_2 + \delta\phi_2$
$\delta(\phi_1 \star \phi_2) = (\delta\phi_1) \star \phi_2 + \phi_1 \star (\delta\phi_2)$

with these two inputs:

$\delta S = 0$
$(\delta(F^{\mu\nu} \star F_{\mu\nu}) = [\alpha, F^{\mu\nu} \star F_{\mu\nu}]_\star)$

NC GAUGE SYMMETRY

- SECOND OPTION
  G.T. are undeformed (comm. space G.T.)
  $\delta_\alpha = -i\alpha\psi$; $\delta_\alpha A_\mu = \partial_\mu \alpha + i(A_\mu \alpha - \alpha A_\mu)$

  Leibniz rule is deformed:

- Reconsider the example of adj. field ϕ:
  $\phi(x) \rightarrow u(x)\phi(x)u^{-1}(x)$
  with $uu^{-1} = u^{-1}u = 1$ (standard inverse).
  $$\phi_1' \star \phi_2' = (u\phi_1 u^{-1}) \star (u\phi_2 u^{-1}) \neq u(\phi_1 \star \phi_2)u^{-1}$$
  $$\Rightarrow \delta(\phi_1 \star \phi_2) \neq (\delta\phi_1) \star \phi_2 + \phi_1 \star (\delta\phi_2)$$
  $$\delta_\alpha(f \star g) = \sum_n \left(-\frac{i}{2}\right)^n \frac{\theta^{\mu_1\nu_1} \ldots \theta^{\mu_n\nu_n}}{n!} (\delta_{\partial_{\mu_1} \ldots \partial_{\mu_n^\alpha}} f \star \partial_{\nu_1} \ldots \partial_{\nu_n} g + \partial_{\mu_1} \ldots \partial_{\mu_n} f \star \delta_{\partial_{\nu_1} \ldots \partial_{\nu_n \alpha}} g)$$

Twisted Leibniz rule and Gauge symmetry

Consider two fields transforming as
$\delta_\alpha \phi_1 = i\alpha\phi_1$, $\delta_\alpha \phi_2 = i\alpha\phi_2$

Twisted comultiplication rule,
$\delta_\alpha(\phi_1 \star \phi_2) = i\mathring{X}_{\alpha^a} \star ((T^a\phi_1) \star \phi_2 + \phi_1 \star (T^a\phi_2))$

'Unstarring' operator:
$\mathring{X}_f = \sum_{n=0}^{\infty} \frac{1}{n!}(-\frac{i}{2})^n \theta^{\rho_1\sigma_1} \dots \theta^{\rho_n\sigma_n}(\partial_{\rho_1} \dots \partial_{\rho_n} f) \star (\partial_{\sigma_1} \dots \partial_{\sigma_n})$

Ex.: $\delta_\alpha \phi_1 = i\alpha\phi_1 = i\mathring{X}_{\alpha^a} \star T^a \phi_1 = i\mathring{X}_\alpha \star \phi_1$
implies $\delta_\alpha(\phi_1 \star \phi_2) \neq \delta_\alpha\phi_1 \star \phi_2 + \phi_1 \star \delta\phi_2$
$= i\alpha\phi_1 \star \phi_2 + i\phi_1 \star \alpha\phi_2$

$$\begin{aligned}\delta_\alpha(F^{\mu\nu} \star F_{\mu\nu}) &= i\mathring{X}_{\alpha^a} \star ([T^a, F^{\mu\nu}] \star F_{\mu\nu} + F^{\mu\nu} \star [T^a, F_{\mu\nu}]) \\ &= i[\alpha, F^{\mu\nu} \star F_{\mu\nu}]\end{aligned}$$

$\Rightarrow \delta\mathcal{L} = \mathrm{Tr}\delta(F^{\mu\nu} \star F_{\mu\nu}) = 0.$

LAGRANGIAN FORMULATION

- Gauge identities and Euler derivatives:
  $S \sim \int F^{\mu\nu,a} F^a_{\mu\nu}$
  $\delta S \sim \int (D_\mu F^{\mu\nu})^a \delta A^a_\nu = \int L^\nu \delta A_\nu$
  L^ν is the Euler derivative.

- Equ. of motion: $L^\nu = 0$
  Gauge invariance $\leftrightarrow$ Gauge identity.
  Take $\delta A_\nu = D_\nu \alpha$
  $\delta S \sim \int L^\nu D_\nu \alpha \sim \int (D_\nu L^\nu)\alpha$
  $= 0$ (off- shell).

- Gauge identity:
  $D_\nu L^\nu = D_\nu D_\mu F^{\mu\nu} = [F_{\mu\nu}, F^{\mu\nu}]$
  $= 0.$

STAR GAUGE INVARIANCE [FIRST OPTION]

- A general action on NC space:
  $S = \int dt \; L = \int d^4x \; \mathcal{L}(q_\alpha(x,t), \partial_i q_\alpha(x,t) \partial_t q_\alpha(x,t))$
- Arbitrary variation:
  $\delta S = \int d^4x \; \delta q^\alpha(x,t) \star L_\alpha(x,t)$,
  L_α: Euler derivatives.
- Gauge transformation:
  $\delta q^\alpha(x,t) = \sum_{s=0}^{n} (-1)^s \int d^3z \frac{\partial^s \eta^b(z,t)}{\partial t^s} \star \rho^{\alpha b}_{(s)}(x,z)$
  $\eta^b(z,t)$: Parameters; $\rho^{\alpha b}_{(s)}(x,z)$: Generators.
  ρ's involve δ- functions and their derivatives.

$\delta S = -\int d^4z\ \eta^a(z,t) \star \Lambda^a(z,t)$

where, $\Lambda^a(z,t) = \sum_{s=0}^{n} \int d^3x \frac{\partial^s}{\partial t^s} (\rho^{\alpha a}_{(s)}(x,z) \star L_\alpha(x,t))$

Invariance of action $\delta S = 0$ implies, $\Lambda^a(z,t) = 0$ identically valid.

(Star gauge identity) (Off-shell)

Presence of this identity signals star gauge invariance.

From this identity, the generator of star gauge transformations can be computed.

EXAMPLE:

$S = \int d^4x [-\frac{1}{4} F^a_{\mu\nu} \star F^{\mu\nu,a} + \bar{\psi} \star (i\gamma^\mu D_\mu \star - m)\psi]$

$\delta S = -\int d^4x (\delta A^a_\mu \star L^{\mu a} + \delta\psi_i \star L_i + \delta\bar{\psi}_i \star L'_i)$

(using $\int A \star B \star C = \int B \star C \star A = \int C \star A \star B$)

Euler derivatives and $\star$ gauge identity

- EULER DERIVATIVE:

$$L^{\mu a} = -(D_\sigma \star F^{\sigma\mu})^a - \psi_j(\gamma^\mu T^a)_{ij} \star \bar{\psi}_i$$

$$L_i = -i\partial_\mu \bar{\psi}_j(\gamma^\mu)_{ji} - \bar{\psi}_j \star (\gamma^\mu A^a_\mu T^a)_{ji} - m\bar{\psi}_i$$

$$L'_i = -i(\gamma^\mu)_{ij}\partial_\mu \psi_j + (\gamma^\mu A^a_\mu T^a)_{ij} \star \psi_j + m\psi_i$$

- Cov. Der. is in adj. rep.:

$$D_\mu \star \zeta = \partial_\mu \zeta + i[A_\mu, \zeta]_\star = \partial_\mu \zeta + i(A_\mu \star \zeta - \zeta \star A_\mu)$$

$$(D_\mu \star \zeta)^a = \partial_\mu \zeta^a - \frac{1}{2} f^{abc}\{A^b_\mu, \zeta^c\}_\star + \frac{i}{2} d^{abc}[A^b_\mu, \zeta^c]_\star$$

$$[T^a, T^b] = if^{abc}T^c; \ \{T^a, T^b\} = d^{abc}T^c$$

Star gauge identity and gauge generator

$\Lambda^a = -(D^\mu \star L_\mu)^a - iT^a_{ij} \star L_i - iT^a_{ji} L'_i \star \bar{\psi}_j = 0$($\theta = 0$ yields the standard gauge identity)

Generator from the identity

- Consider the A_0 field:

$\Lambda^a(z,t) = \sum_{s=0}^{n} \int d^3x \frac{\partial^s}{\partial t^s}(\rho^{b\alpha a}_{(s)}(x,z) \star L^b_\alpha(x,t))$

(general structure)

$\Lambda^a|_{L_0} = \sum_s \int d^3x \frac{\partial^s}{\partial t^s}(\rho^{b0a}_{(s)}(x,z) \star L^b_0(x,t))$

$= \int d^3x (\rho^{b0a}_{(0)} \star L^b_0 + \rho^{b0a}_{(1)} \star \frac{\partial L^b_0}{\partial t})$

(only $s = 0, 1$ contribute)

Re-express Λ^a in this form.

Generator of G.T.

$$\Lambda^a(z,t)|_{L_0} = -(D^0 \star L_0)^a$$
$$= -\frac{\partial L_0^a}{\partial t} - \frac{1}{2} f^{abc}\{L_0^b, A^{0,c}\}_\star + \frac{i}{2} d^{abc}[L_0^b, A^{0,c}]_\star$$
$$= -\int d^3x \delta^{ab} \delta^3(x-z) \star \frac{\partial L_0^b(x)}{\partial t}$$
$$- \int d^3x [\frac{1}{2}(f^{abc}\{\delta^3(x-z), A^{0,c}(x)\}_\star$$
$$+ i d^{abc}[\delta^3(x-z), A^{0,c}(x)]_\star) \star L_0^b(x)]$$

- Comparison yields:

$$\rho_{(0)}^{b0a}(x,z) = -\frac{1}{2} f^{abc}\{\delta^3(x-z), A_0^c(x)\}_\star - \frac{i}{2} d^{abc}[\delta^3(x-z), A_0^c(x)]_\star$$

$$\rho_{(1)}^{b0a}(x,z) = -\delta^{ab}\delta^3(x-z)$$

Star Gauge transformation

$$\delta q^{\alpha}(x,t) = \sum_{s=0}^{n}(-1)^s \int d^3z \frac{\partial^s \eta^b(z,t)}{\partial t^s} \star \rho^{\alpha b}_{(s)}(x,z)$$

For the A_0 field,

$$\delta A^{0a}(x,t) = \sum_{s}(-1)^s \int d^3z \frac{\partial^s \eta^b(z,t)}{\partial t^s} \star \rho^{a0b}_{(s)}(x,z)$$

$$= \int d^3z(\eta^b(z,t) \star \rho^{a0b}_{(0)}(x,z) - \frac{\partial \eta^b}{\partial t}(z,t) \star \rho^{a0b}_{(1)}(x,z))$$

(Only $s = 0, 1$ contribute)

Substitute the expression for $\rho_{(0)}, \rho_{(1)}$
(Use $A(x) \star \delta(x-z) = \delta(x-z) \star A(z)$)

$$\delta A^{0a} = \partial^0 \eta^a - \frac{1}{2} f^{abc}\{A^{0b}, \eta^c\}_\star + \frac{i}{2} d^{abc}[A^{0b}, \eta^c]_\star$$

$$= (D^0 \star \eta)^a$$

(Reproduces the well known star deformed G.T.)

TWISTED GAUGE SYMMETRY

$\mathcal{L} = -\frac{1}{4}\int d^4x \mathrm{Tr}(F_{\mu\nu} \star F^{\mu\nu})$

- Standard G.T.

  $\delta A_\mu = \partial_\mu \eta + i[A_\mu, \eta] = \partial_\mu \eta + i(A_\mu \eta - \eta A_\mu)$

  $\delta F_{\mu\nu} = -i[\eta, F_{\mu\nu}]$

  $\delta(F_{\mu\nu} \star F^{\mu\nu}) = -i[\eta, F_{\mu\nu} \star F^{\mu\nu}]$ (already shown)

  Equation of motion and Euler derivatives are identical to the previous case.

  $\delta S \sim \int \delta A_\mu \star L^\mu$

  $\Rightarrow L_\mu = -D^\sigma \star F_{\sigma\mu}$

  Gauge identity: $D^\mu \star L_\mu = 0 \rightarrow$ (Off shell)

  and

  Field equ. $L_\mu = 0$

- <u>GAUGE GENERATOR</u>

$$\delta A_0^a(x) = \sum_s (-1)^s \int d^3z \frac{\partial^s \eta^b}{\partial t^s}(z,t)\rho^{a0b}_{(s)}(x,z)$$

$$= \int d^3z \eta^b(z,t)\rho^{a0b}_{(0)}(x,z) - \int d^3z \frac{\partial \eta^b}{\partial t}(z,t)\rho^{aob}_{(1)}(x,z)$$

A simple comparison yields:

$$\rho^{a0b}_{(0)}(x,z) = f^{abc}A_0^c\delta^3(x-z)$$

$$\rho^{a0b}_{(1)}(x,z) = -\delta^{ab}\delta^3(x-z)$$

(Note the absence of $\star$ products in ρ)

Relation connecting gauge identity (Λ^a) with the generator gets deformed.

$$\Lambda^a(z,t) = \sum_{s=0}^n \int d^3x \frac{\partial^s}{\partial t^s}(\rho'^{b\mu a}_{(s)}(x,z)L^b_\mu(x,t))$$

$\rho' = \rho + \theta$- deformed terms.

HAMILTONIAN FORMULATION

G.T. are obtd by Poisson bracketting with the Generator. Also follow usual Leibniz rule.

$G = \int \epsilon_a(y)\phi_a(y)dy$; $\epsilon_a \rightarrow$ Parameters, $\phi_a \rightarrow$ F.C.C.

$\delta F(x) = \int dy \epsilon_a(y)\{F(x), \phi_a(y)\}_{PB}$

- Alternative def.
  $\delta F(x) = \int dy\{F(x), \epsilon_a(y)\phi_a(y)\}_{PB}$
- Two differ by trivial G.T.
  $\delta S = \frac{\partial S}{\partial q_i}\delta q_i;$ $\quad \delta q_i = \Lambda_{ij}\frac{\partial S}{\partial q_j}$
  $= 0$ $\quad \Lambda_{ij} = -\Lambda_{ji}$
  (Trivial gauge symmetry)

GENERAL FORMULATION

A gauge theory (First class) is characterised by the following closed algebra:

$\{H, \phi_a(x)\} = \int dy V_a^b(x,y) \star \phi_b(y)$

$\{\phi_a(x), \phi_b(y)\} = \int dy' C_{ab}^d(x,y,y') \star \phi_d(y')$

H: Hamiltonian; ϕ_a: F.C.C.

- Equation of motion:

  $H_T = H_C + \int dx \; v^{a_1}(x) \star \phi_{a_1}(x)$

  $\dot{q}_i(x) = \{q_i, H_T\} = \{q_i, H_c\} + \int dy \; v^{a_1}(y) \star \{q_i(x), \phi_{a_1}(y)\}$

  $(\phi_{a_1} \approx 0)$ weak imposition.

  v_{a_1}: Lag. multipliers enforcing the Primary F.C.C. (ϕ_{a_1}).

DIRAC'S ALGORITHM (NC SYSTEM)

Generator of G.T. $= G = \int dx \; \epsilon^a(x) \star \phi_a(x)$

Variation: $\delta F(x) = \int dy \; \epsilon^a(y) \star \{F(x), \phi_a(y)\}$

Not all the ϵ^a's are independent.

Independent parameters equal the no. of independent Primary F.C.C. (= no. of gauge identities in Lag. formulation)

Primary constraints: ϕ_{a_1}: Follow directly from the Lagrangian.

Secondary constraints: ϕ_{a_2}: Follow as consistency requirements $\{\phi_{a_1}, H_T\} \approx 0$

(Equations of motion expressed in terms of phase space variables)

Parametric conditions

Derivation of E.L. eq. from action principle requires,

$\delta \frac{d}{dt} q_i = \frac{d}{dt} \delta q_i$

Impose this at the Hamiltonian level:
Constraints are Irreducible (independent) / No S.C.C.

- Yields two Conditions:

$$\delta v^{b_1}(x) = \frac{d}{dt}\epsilon^{b_1}(x) - \int dy \epsilon^a(y) \star V_a^{b_1}(y,x) - \int dydz\ \epsilon^a(y) \star v^{a_1}(z) \star C_{a_1 a}^{b_1}(z,y,x)$$

$$\frac{d}{dt}\epsilon^{b_2}(x) = \int dy\ \epsilon^a(y) \star V_a^{b_2}(y,x) + \int dydz\ \epsilon^a(y) \star v^{a_1}(z) \star C_{a_1 a}^{b_2}(z,y,x)$$

Solving these 'b_2' equations, one obtains the independent ϵ^{b_1} to be used in the generator of G.T.

NCYM theory (usual coproduct)

$S = -\frac{1}{4} \int dx \ Tr \ F_{\mu\nu} \star F^{\mu\nu}$

To find the generator of G.T.

- Primary Constraint:

  $\Pi_0^a = \frac{\partial \mathcal{L}}{\partial \dot{A}^{\theta_1 a}} \approx 0 \qquad = \phi_1^a$

- Secondary Constraint:

  $H_c = \int \left[\frac{1}{2} \Pi^{ia} \star \Pi^{ia} + \frac{1}{4} \ F_{ij}^a \star F_{ij}^a + A_0^a \star (D_i \star \Pi_i)^a \right]$

  $\{\Pi_0^a, H_T\} \approx 0$

  $\Rightarrow \ D_i \star \Pi_i \approx 0 \qquad = \phi_2$ (NC GAUSS)

- Involutive algebra

  $\{\phi_1, \phi_1\} = \{\phi_1, \phi_2\} = 0$

  $\left\{\phi_2^a(x), \phi_2^b(y)\right\} = \frac{1}{2} f^{abc} \left\{\delta(x-y), \phi_2^c(x)\right\}_\star - \frac{i}{2} d^{abc} \left[\delta(x-y), \phi_2^c(x)\right]_\star$

  $\{H, \phi_1^a\} = \phi_2^a$

  $\{H, \phi_2^a\} = \frac{1}{2} f^{abc} \left\{A^{0b}, \phi_2^c\right\}_\star - \frac{i}{2} d^{abc} \left[A^{0b}, \phi_2^c\right]_\star$

FIXING PARAMETERS

$$\frac{d\epsilon^{b_2}(x)}{dt} = \int dy\ \epsilon^{a}(y) \star V_a^{b_2}(y,x)$$

$$\left(V_1^2\right)^{ab}(x,y) = \delta^{ab}\delta(x-y)$$

$$\left(V_2^2\right)^{ab} = -\frac{1}{2}f^{abc}\left\{\delta(x-y), A^{0c}(y)\right\}_\star - \frac{i}{2}d^{abc}\left[\delta(x-y), A^{0c}(y)\right]_\star$$

Insert in the above condition;

$$\frac{d}{dt}\epsilon^{2a}(x) = \int dy\ \epsilon^{1b}(y) \star \left(V_1^2\right)^{ba}(y,x) + \int dy\ \epsilon^{2b}(y)\left(V_2^2\right)^{ba}(y,x)$$

$$= \epsilon^{1a}(x) + \frac{1}{2}f^{abc}\left\{\epsilon^{2b}(x), A^{0c}(x)\right\}_\star - \frac{i}{2}d^{abc}\left[\epsilon^{2b}, A^{0c}\right]_\star$$

$$\Rightarrow \quad \epsilon^{1a} = \left(D_0 \star \epsilon^2\right)^a$$

$$\epsilon^1 = D_0 \star \epsilon^2$$

leads to only one independant parameter.

Gauge Transformations

$$G = \int dx\ \epsilon^a \star \phi_a = \int dx\ \left(\epsilon^{1a} \star \phi_1^a + \epsilon^{2a} \star \phi_2^a\right)$$
$$= \int dx\ \left[\left(D_0 \star \epsilon^2\right)^a \star \Pi_0^a + \epsilon^{2a} \star (D_i \star \Pi_i)^a\right]$$

$$\delta A_0^a(x) = \int dy\ \left(D_0 \star \epsilon^2\right)^b (y) \star \left\{A_0^a(x), \Pi_0^b(y)\right\}$$
$$= \left(D_0 \star \epsilon^2\right)^a (x)$$

$$\delta A_i^a(x) = \int dy\ \epsilon^{2b} \star \left\{A_i^a(x), (D_j \star \Pi_j)^b (y)\right\}$$

$$\left(\ \{A, B \star C\} = \{A, B\} \star C + B \star \{A, C\}\ \right)$$

$$\delta A_i^a = \left(D_i \star \epsilon^2\right)^a$$

- Combining both expressions:

$$\delta A_\mu = D_\mu \star \epsilon^2 \qquad \text{star deformed G.T.}$$

TWISTED GAUGE SYMMETRY

- A simple interpretation for the twisted coproduct:
  $\delta_\eta(\phi \star \psi) = i\eta^a \{(T^a\phi) \star \psi + \phi \star (T^a\psi)\}$
  if $\delta_\eta\phi = i\eta\phi,\ \delta_\eta\psi = i\eta\psi$
- Take the usual coproduct and put the gauge parameter outside the $\star$ operation at the end:
  $$\begin{aligned}\delta_\eta(\phi \star \psi) &\cong (\delta_\eta\phi) \star \psi + \phi \star \delta_\eta\psi \\ &= (i\eta\phi) \star \psi + \phi \star (i\eta\psi) \\ &= i\eta^a (T^a\phi) \star \psi + i\eta^a\phi \star (T^a\psi)\end{aligned}$$
- Product rule is identical to the undeformed case:
  $$\begin{aligned}\delta_\eta(\phi\psi) &= (\delta_\eta\phi)\,\psi + \phi\,(\delta_\eta\psi) \\ &= i\eta^a \{(T^a\phi)\,\psi + \phi\,(T^a\psi)\}\end{aligned}$$

Gauge Transformations

$G = \int dx \; [(D_0 \star \eta)^a \star \Pi_0^a + \eta^a \star (D_i \star \Pi_i)^a]$

Take the P.B. and push η^a outside the $\star$ operation at the end.

$$\begin{aligned}\delta_\eta A_0^a(x) &= \int dy \; (D_0 \star \eta)^b (y) \star \left\{A_0^a(x), \Pi_0^b(y)\right\} \\ &\sim \int dy \; (D_0 \star \eta)^b (y) \; \delta^{ab} \star \delta(x-y) \\ &\sim \int dy \; \Big(\partial_0 \eta^a - \tfrac{g}{2} f^{abc} \left\{A_0^b, \eta^c\right\}_\star \\ &\qquad\qquad + i\tfrac{g}{2} d^{abc} \left[A_0^b, \eta^c\right]_\star (y) \star \delta(x-y) \\ &= \partial_0 \eta^a - g \; f^{abc} A_0^b \eta^c = (D_0 \eta)^a (x)\end{aligned}$$

$\Longrightarrow$ The undeformed transformation law.

For the other variables (A_i^a, ψ etc.) the same analysis holds.
Also, the product rule can be verified.

Comparison with the Hopf algebraic approach

Generator (Sch. rep.) as in undeformed situation:

$$G_\alpha = \int dz \left(\partial_\mu \alpha^l(z) + g\ \alpha^r(z)\ A^s_\mu(z)\ f^{rsl}\right)\ \frac{\delta}{\delta A^l_\mu(z)}$$

$$\implies [G_\alpha, G_\beta] = ig\ G_{[\alpha,\beta]}$$

Twisted coproduct is obtained by applying this G_α to the usual (primitive) coproduct.

This G_α however only generates the standard G.T.

Star G.T. are generated by our G which satisfies,

$$[G_\alpha, G_\beta] = G_{[\alpha,\beta]_\star} \qquad [\alpha,\beta]_\star = \alpha\star\beta - \beta\star\alpha$$

Calculation of P.B. yields either the twisted case or the usual one.

FINAL REMARKS

- Both types of gauge symmetry in NC gauge theories can be understood in either Lagrangian or Hamiltonian formulations.
- The $\theta \to 0$ limit is smooth in both cases leading to the commutative space expressions.
- This analysis holds only for constant θ. For other types of 'θ', implementation of twisted gauge symmetry is an open issue.
- Could be useful in the study of twisted diffeos.

Thank You