NONCOMMUTATIVE INSTANTON BUNDLES AND INDEX PAIRING

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Abstract

I overview how to reduce the index pairing for a suspension ΣA of algebra A to the index pairing for A itself.
This is applied to quantum instanton bundles of arbitrary charges over non-commutative deformations of the sphere.
Based on arXiv:math/0702001 with T. Hadfield, P. M. Hajac, R. Matthes.

I'll review and put in general context a number of recent index computations for noncommutative instanton bundles.

Classically: vector bundle $\longrightarrow$ conection $\longrightarrow$ curvature $\longrightarrow$ Ch $\longrightarrow$ $\#$

that depends only on the equivalence class of the bundle, i.e. its K_0 class†.

In NCG a topological space, as motivated by Gelfand-Naimark, corresponds to a C*-algebra, while a topological vector bundle, as motivated by Serre-Swan, corresponds to a finitely-generated projective module (over C*-algebra of the underlying noncommutative space).

The kind of C*-algrebras discussed here is suspension ΣA of another C*-algebra A.

The K_0-classes of are certain equiv. classes of projectors (idempotents suffice).

I'll focus on computing appropriate index pairings.

† no fear, cf. string th.

For that, employ the Mayer-Vietoris 6-term exact sequences for K-theory and K-homology and take advantage of the compatibility of the index pairing $< \, , \, >$ with the connecting maps [HR, BM]

$$\partial : K_1(A) \longrightarrow K_0(\Sigma A) \quad \text{and} \quad \delta : K^0(\Sigma A) \longrightarrow K^1(A).$$

This considerably simplifies computations by reducing them from ΣA to A.

Now $K_1(A)$ consists of (equiv. classes of) unitaries $u \in M_N(A)$ (invertible suffice). I'll give ∂ [Ba, GH04] by an explicit expression for an idempotent p in the matrix algebra $M_{2N}(\Sigma A)$ in terms of u.

Of particular interest is the case $K_1(A) = \mathbb{Z}$. Then I take u whose class generates $K_1(A)$, and consider idempotents p_n $(n \in \mathbb{Z})$ entering the definition of $\partial[u^n]$. $(\Sigma A)^{2N} p_n$ is a finitely-generated projective left ΣA-module (of sections) of our non-commutative vector bundle.
For $A = C(S^3)$, $\Sigma A = C(S^4)$ this yields the module of continuous sections of the classical instanton vector bundle of charge n over $S^4 = \Sigma S^3$ [At, pp. 14-27].

This formalism serves as a common denominator for both the $SU_q(2)$ and θ-deformed instanton bundles, and computes the index pairings for all instanton idempotents p_n.
The former construction [Pfl, DLM] is based on the non-reduced suspension of the quantum $SU(2)$ group [Wo87], whereas the latter [CL] is based on the non-reduced suspension of the θ-deformation of S^3 [Mat].

The (non-reduced) suspension

$$\Sigma A := \{ \ f \in C([0,1],A) \mid f(0), f(1) \in \mathbb{C} \ \}$$

can be identified with the fiber product (glueing) over A of cones of A

$$B_0 = \{ \ f \in C([0,\frac{1}{2}],A) \mid f(0) \in \mathbb{C} \ \}, \qquad B_1 = \{ \ f \in C([\frac{1}{2},1],A) \mid f(1) \in \mathbb{C} \ \}$$

via

$$\Sigma A \ni f \mapsto (f|_{[0,\frac{1}{2}]}, f|_{[\frac{1}{2},1]}) \in B_0 \times_A B_1 \subset B_0 \oplus B_1 \ . \qquad (.1)$$

Fig.

K-theory of ΣA can be described by the Mayer-Vietoris 6-term exact cyclic sequence, but since cones are contractible,

$$K_0(B_i) = \mathbb{Z}, \quad K_1(B_i) = 0$$

it reduces to the exact sequence

$$0 \longrightarrow K_1(A) \xrightarrow{\partial} K_0(\Sigma A) \xrightarrow{r_{0*} \oplus r_{1*}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\pi_{1*} - \pi_{0*}} K_0(A) \longrightarrow K_1(\Sigma A) \longrightarrow 0 \ ,$$

where $r_i : \Sigma A \to B_i$ are the restrictions and $\pi_i : B_i \to A$ are both given by $f \mapsto f(\frac{1}{2})$.

Assume that $K_0(A) = \mathbb{Z}$ with generator $[1]$, where 1 is the unit of A, and that $K_1(A) = \mathbb{Z}$ with generator denoted by $[u]$. It follows that

$$K_0(B_0) \oplus K_0(B_1) \overset{\pi_{1*} - \pi_{0*}}{\longrightarrow} K_0(A)$$

is surjective, and thus $K_1(\Sigma A) = 0$.

Moreover $K_0(\Sigma A) = \mathbb{Z}^2$, with one generator given by $[I]$, where $I(t) = 1$.

Next, the map $\partial : K_1(A) \to K_0(\Sigma A)$ is injective and $[I] \notin \mathrm{Im}(\partial)$.

Thus the class $\partial[u]$ is the second generator of $K_0(\Sigma A)$.

The map ∂ can be given explicitly.

Let $u \in GL_N(A)$ (invertible matrix).
Let $\psi \in C([1/2, 1])$ s.t. $\psi(1/2) = 1$ and $\psi(1) = 0$. (Later further restricted).
Theorem .1. *The connecting homomorphism* $\partial : K_1(A) \to K_0(\Sigma A)$ *reads*

$$\partial[u] = [p] - [1_N] \ , \qquad (.2)$$

where the idempotent $p \in M_{2N}(\Sigma A)$ *is the product* $p = X\,Y$ *of a column* $\times$ *row, of* A*-valued functions of* $t \in [0, 1]$

$$X = \begin{cases} \begin{bmatrix} 1_N \\ 0_N \end{bmatrix} & if \quad t \in [0, 1/2] \\ \begin{bmatrix} \psi(2-\psi^2)\,1_N \\ (1-\psi^2)u^{-1} \end{bmatrix} & if \quad t \in [1/2, 1] \end{cases} ,$$

$$Y = \begin{cases} [1_N, 0_N] & if \quad t \in [0, 1/2] \\ [\psi\,1_N, (1-\psi^2)u] & if \quad t \in [1/2, 1] \end{cases} .$$

It can be seen that the components of X and Y are not in ΣA but those of p are (!)
Moreover, $Y X = I$ (constant function $I(t) = 1$), and so $p^2 = p$.

I mention that for the (reduced) suspension of A,

$$SA := \{\, f \in C([0,1], A) \mid f(0) = f(1) = 0\} \tag{.3}$$

there is another method of constructing a projection $q \in M_{2N}(SA)$ in terms of a unitary $u \in M_N(A)$ [W-O] p.139.
It works as well for the non-reduced suspension ΣA of A and yields

$$q = TT^{\dagger}, \quad \text{where} \quad T = \left[\begin{array}{c} \cos^2(\pi t/2) - \sin^2(\pi t/2)\, u \\ \cos(\pi t/2)\sin(\pi t/2)(u^{\dagger} - 1) \end{array} \right] \tag{.4}$$

($^{\dagger} \; = \; {}^{t} \circ *$).

This projection (for $N = 2$) was related to the (complex) Bott projector in [W-O] p.145.

But the Milnor-Bass projection is much more suitable for us:

• It works equally well for both the algebraic [Ba] and C*-algebraic K_1 (e.g., see [GH04]).

• More importantly, it is compatible (!) with the pairing $< \cdot, \cdot >$ of $K_\bullet$ with $K^\bullet$, i.e the connecting maps ∂ and δ are transposed one of the other.

Now, by 'universal coefficient theorem', if the K-groups K_j are free, then $K^j \cong K_j$. Thus, in our situation

$$K^0(A) = \mathbb{Z}, \quad K^0(B_i) = \mathbb{Z}, \quad K^0(\Sigma A) = \mathbb{Z}^2 \ ,$$

$$K^1(A) = \mathbb{Z}, \quad K^1(B_i) = 0, \quad K^1(\Sigma A) = 0$$

and $\delta : K^0(\Sigma A) \to K^1(A)$ is surjective.

Now, use the Milnor formula the n-th power of the generator $[u]$ of $K_1(A) = \mathbb{Z}$, and obtain an explicit idempotent p_n over ΣA.
The compatibility of $< \cdot, \cdot >$ with (.2), means that the pairing of p_n with a preimage $\mathbf{w} \in K^0(B)$ of the generator 1 of $K^1(A) \cong \mathbb{Z}$ under the surjective map δ is

$$< [p_n], \mathbf{w} >=< \partial[v^n], \mathbf{w} >=< [v^n], \delta(\mathbf{w}) >= n < [v], 1 > .$$

(Actually also $< [1], \mathbf{w} >= 0$ used here).

Now, use the Milnor formula the n-th power of the generator $[u]$ of $K_1(A) = \mathbb{Z}$, and obtain an explicit idempotent p_n over ΣA.
The compatibility of $< \cdot, \cdot >$ with (.2), means that the pairing of p_n with a preimage $\mathbf{w} \in K^0(B)$ of the generator $\mathbf{z}$ of $K^1(A) \cong \mathbb{Z}$ under the surjective map δ is

$$< [p_n], \mathbf{w} >=< \partial[v^n], \mathbf{w} >=< [v^n], \delta(\mathbf{w}) >= n < [v], \mathbf{z} > . \qquad (.5)$$

(Actually $< [1], \mathbf{w} >= 0$ also used here).

Thus we reduce the work of Jack (LHS), Jane (RHS) and Jim ($=$), to just 1 task of 3!

Recall that $K^1(A)$ consists of (equivalence classes of) Fredholm modules (A, F, H).
In the rest of my talk the task will be, for certain concrete and natural C*-algebras, to find and analyze unitaries u and Fredholm modules z, s.t. $\mathbf{z} = [z]$.

The classical instanton vector bundle E_n with charge $n \in \mathbb{Z}$.

The base manifold is S^4 with a covering consisting of two open 4-discs U_0 and U_1. Since $U_0 \cap U_1$ is homotopic to S^3, we can replace U_0 and U_1 by the closed 4-discs D^4 (cones of S^3), and view S^4 as two D^4 glued along their boundary 3-spheres: $D^4 \coprod_{S^3} D^4$. In the dual language: the fiber product of algebras of continuous functions.

The bundle E_n is obtained by glueing $U_0 \times \mathbb{C}^2$ and $U_1 \times \mathbb{C}^2$ over $U_0 \cap U_1$ using the transition function v_n with winding number $n \in \pi_3(S^3) \cong \mathbb{Z}$ (see [At, p. 15]). Explicitly, $v_n : S^3 \to SU(2)$ can be described as follows. Let

$$x = (x_0, x_1, x_2, x_3) \longmapsto \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix}, \qquad (.6)$$

where $\alpha = x_0 + ix_3$, $\beta = x_1 + ix_2$, be the well known identification of $\mathbb{R}^4$ as a subalgebra of $M_2(\mathbb{C})$, and (after a restriction) of S^3 with $SU(2)$. Then, v_n is the n-th (matrix) power of (.6). Clearly v_n can be also viewed as a unitary element in $M_2(C(S^3))$.

The continuous sections of E_n form a projective module $C(E_n)$ over $C(S^4)$, namely

$$C(E_n) \simeq (C(S^4))^4 \, p_n,$$

where the projection $p_n \in M_4(C(S^4))$ is built [Kar] from the transition functions

$$\phi_{00} = \phi_{11} = 1, \ \phi_{10} = v_n, \ \phi_{01} = v_n^\dagger$$

and a partition of unity f_0, f_1, subordinated to the covering U_0, U_1, as follows

$$p_n = \begin{bmatrix} f_0 & \sqrt{f_0 f_1} v_n^\dagger \\ \sqrt{f_0 f_1} v_n & f_1 \end{bmatrix} = LL^\dagger, \ \text{ where } \ L = \begin{bmatrix} \sqrt{f_0} \\ \sqrt{f_1} v_n \end{bmatrix}. \tag{.7}$$

Using another identification of $\mathbb{R}^4$ with the quaternions $\mathbb{H}$, S^3 becomes identified with

$$\mathbb{H}_1 = \{g \in \mathbb{H} \mid \bar{g}g = 1\},$$

where $^-$ is quaternionic conjugation, and v_n becomes the map raising $h \mapsto h^n$, $h \in \mathbb{H}$. (When composed with the inverse of (.6), S^3 becomes identified with $SU(2)$).

Corresponding to well-known homeomorphisms $S^4 = \Sigma S^3 = \mathbb{H}P^1$, there are three 'coordinate systems':

$$(\tau, h), \quad \tau \in \mathbb{R},\ h \in \mathbb{H} \equiv \mathbb{R}^4, \quad \text{with } \tau^2 + \bar{h}h = 1 \qquad \text{(spherical)}; \tag{.8}$$

$$(t, g), \quad t \in [0,1],\ g \in \mathbb{H}_1 \equiv S^3 \qquad \text{(suspension)}; \tag{.9}$$

$$z, \quad z \in \mathbb{R}^4 \cup \{\infty\} \equiv \mathbb{H} \cup \{\infty\} \qquad \text{(stereographic)}. \tag{.10}$$

They are related to each other as follows:

$$g = \frac{h}{\sqrt{1-\tau^2}},\ t = \frac{1+\tau}{2}; \tag{.11}$$

$$z = \sqrt{\frac{t}{1-t}}\,g; \quad z = \frac{h}{1-\tau}\ . \tag{.12}$$

In these coordinates the *tautological* projection in $M_4(C(\mathbb{H}P^1))$ (that at $[l] \in \mathbb{H}P^1$ projects on the quaternionic line l in $\mathbb{H}^2$) becomes respectively

$$\frac{1}{2}\begin{bmatrix} 1-\tau & \bar{h} \\ h & 1+\tau \end{bmatrix} = HH^\dagger, \quad \text{where} \quad H = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{1-\tau} \\ \frac{h}{\sqrt{1-\tau}} \end{bmatrix}, \tag{.13}$$

$$\begin{bmatrix} 1-t & \sqrt{t(1-t)}\bar{g} \\ \sqrt{t(1-t)}g & t \end{bmatrix} = GG^\dagger, \quad \text{where} \quad G = \begin{bmatrix} \sqrt{(1-t)} \\ \sqrt{t}g \end{bmatrix}, \tag{.14}$$

$$\frac{1}{z\bar{z}}\begin{bmatrix} 1 & \bar{z} \\ z & z\bar{z} \end{bmatrix} = ZZ^\dagger, \quad \text{where} \quad Z = \frac{1}{\sqrt{1+z\bar{z}}}\begin{bmatrix} 1 \\ z \end{bmatrix}. \tag{.15}$$

Note that when $g = v_n$, $GG^\dagger = p_n = LL^\dagger$ of (.7) (choose $f_0 = 1-t$ and $f_1 = t$). Thus all these are equivalent forms of the instanton projection.

Now I'll relate them to the projection of Theorem .1.

Proposition .2. *Let $g \in U_2(C(S^3))$ be a unitary matrix and $GG^\dagger$ be a projection as above, with G given in (.14). Let $a \in GL_2(C(S^3))$ be an invertible matrix and $X\,Y$ an idempotent as in Theorem .1. If the K_1-class of g equals the K_1-class of a^{-1} (i.e. g and a^{-1} are homotopic), then the K_0-class of $GG^\dagger$ equals the K_0-class of $X\,Y$.*

Pf. W.l.g. choose $\psi = \sqrt{1-t}$ in Theorem .1. We exhibit a homotopy between $X\,Y$ and $GG^\dagger$ as a composition of three homotopies. The first one is just from a^{-1} to g. The second one 'expands' the interval $[1/2, 1]$ to $[0, 1]$, i.e. for $s \in [0, 1]$ define a family of X^s, Y^s as in Theorem .1, with ψ substituted by $\psi^s \in C([\frac{s}{2}, 1])$, where $\psi^s(t) := \psi(\frac{t+1-s}{2-s})$. (Note that $\psi^s(\frac{s}{2}) = 1, \psi^s(1) = 0$.) Obviously, $\psi^1 = \psi$. Due to $Y^s X^s = 1$, $X^s Y^s$ is an idempotent $\forall s \in [0, 1]$, and gives a homotopy between $X\,Y$ and $X^0 Y^0$.
Next, note that for arbitrary $V = [v_1, v_2]^T$ and $W = [w_1, w_2]$, with $v_1, v_2, w_1, w_2 \in A$, if WV is invertible in A then $V\,(WV)^{-1}W$ is an idempotent in $M_2(A)$. Take now $V = rX^0 + (1-r)G$ and $W = rY^0 + (1-r)G^\dagger$, where $r \in [0, 1]$. We have $WV = \kappa r^2 - \kappa r + 1$, where $\kappa = t(\sqrt{t} - 1)^2$. Clearly $\kappa \in [0, 1]$. If $\kappa = 0$, $WV = 1$, and if $\kappa \neq 0$, then WV is a second order polynomial in r with the discriminant $\kappa(\kappa - 4) < 0$. Hence, $\forall r \in [0, 1]$, WV is invertible and $V\,(WV)^{-1}W$ defines a homotopy via projections between $X^0 Y^0$ and $GG^\dagger$. By the composition of these homotopies, we get the equivalence of the projections as stated. $\square$

As a consequence the Milnor, Karoubi, spherical, suspension and stereographic are five equivalent projections.

This proposition (& Pf) remain valid if we substitute $C(S^3)$ by any unital C*-algebra.
In particular, for certain noncommutative deformations A of $C(S^3)$ and $C(S^4)$.
Also what I have said above about different charts etc. remains valid.
I'll present now such noncommutative examples that satisfiy our assumptions on K-groups, and a noncommutative deformation of E_n (in the dual language).

Take A to be the universal unital C*-algebra generated by α, β with relations (c.f. [D])

$$\alpha\beta = q\beta\alpha\,,\ \alpha\beta^* = \bar{q}\beta^*\alpha\,,\ \beta\beta^* = \beta^*\beta\,,\ \alpha^*\alpha + \beta\beta^* = 1\,,\ \alpha\alpha^* + |q|^2\beta\beta^* = 1\,, \tag{.16}$$

where $q \in \mathbb{C}$ and (without loss of generality) $|q| \leq 1$.
The case $q = 1$ is just (isomorphic to) the commutative unital C*-algebra $C(S^3)$.
The case $q \in \mathbb{R}$ underlies Woronowicz's quantum group $C(SU_q(2))$ [Wo87].

The quantum $SU(2)$ case

We discuss first the case $q \in \mathbb{C}$ with $0 < |q| < 1$ and denote A by $C(S_q^3)$.
It turns out that all the C*-algebras $C(S_q^3)$ are isomorpic [P. Soltan].
There is a faithful *-representation π of $C(S_q^3)$ on the Hilbert space $\ell^2(\mathbb{N} \times \mathbb{Z})$, with basis $\{e_{m,n}\}$, given by

$$\pi(\alpha)e_{m,n} = (1 - |q|^{2m})^{1/2}\, e_{m-1,n}, \quad \pi(\beta)e_{m,n} = q^m e_{m,n+1}.$$

(I identify $C(S_q^3)$ with $\pi(C(S_q^3))$).
From [MNW90] it follows that both K-groups of $C(S_q^3)$ are $\mathbb{Z}$ and the generator of K_0 given by [1] (thus our assumptions are satisfied).
Moreover, $K_1(C(S_q^3))$ is generated by the unitary

$$u\, e_{m,n} = \begin{cases} e_{0,n+1} & : \ m = 0 \\ e_{m,n} & : \ m \neq 0 \end{cases} . \tag{.17}$$

The K^1 homology is also $\mathbb{Z}$, generated by the class of the Fredholm module

$$z = (\ell^2(\mathbb{N} \times \mathbb{Z}), \pi, F)\ , \quad \text{where} \quad F e_{m,n} = \mathrm{sign}(n) e_{m,n}. \tag{.18}$$

The formulae given in [Co94] express the pairing as

$$< [u], \mathbf{z} >= \lim_{k\to\infty} (-)^k \, 2^{-(2k+1)} \, \mathrm{Tr}((u^\dagger - 1)[F,u]([F,u^\dagger][F,u])^k) \quad (.19)$$

which it is not difficult to compute to be 1.
Instead of this u we want to employ an invertible element of $M_2(C(S_q^3))$ that is a q-deformation of the classical generator (.6)

$$v = \begin{bmatrix} \alpha & -\bar{q}\beta^* \\ \beta & \alpha^* \end{bmatrix} \in M_2(C(S_q^3)) \ . \quad (.20)$$

That (invertible) v is in fact unitary, follows from (.16).
The class $[v]$ was claimed in [Co] to be nontrivial in $K_1(C(S_q^3))$ and its pairing with the K-homology class of the spectral triple of [CP] was left as an exercise.
In [vSDLSV] the (local) index pairing of v with the unbounded K-cycle (3-summable spectral triple) constructed in [DLSvSV] was computed to be 1.
Since by Connes' index theorem [Co94] p. 296 this pairing is always integer, so $[v]$ also must generate $K_1 \cong \mathbb{Z}$ and in fact $[v] = [u]$.

We show this explicitly in a simpler way by pairing v with the same $\mathbf{z} \in K_1$ given by (.18) (F extends to $\ell^2(\mathbb{N} \times \mathbb{Z}) \otimes \mathbb{C}^2$ as $\tilde{F} = F \oplus F$). Since $[\tilde{F}, \alpha] = 0 = [\tilde{F}, \alpha^*]$,

$$[\tilde{F}, v^\dagger][\tilde{F}, v] = \begin{bmatrix} [F, \alpha^*] & [F, \beta^*] \\ -q[F, \beta] & [F, \alpha] \end{bmatrix} \begin{bmatrix} [F, \alpha] & -\bar{q}[F, \beta^*] \\ [F, \beta] & [F, \alpha^*] \end{bmatrix} = \begin{bmatrix} f^* f & 0 \\ 0 & -|q|^2 f f^* \end{bmatrix}$$

where $f = [F, \beta]$, $f^* = [F, \beta]^* = -[F, \beta^*]$. Hence

$$(v^\dagger - 1)[\tilde{F}, v]([\tilde{F}, v^\dagger][\tilde{F}, v])^k = (-)^k \begin{bmatrix} \beta^* f (f^* f)^k & \bar{q}|q|^{2k}(\alpha^* - 1) f^* (f f^*)^k \\ (\alpha - 1) f (f^* f)^k & -|q|^{2k+2} \beta f^* (f f^*)^k \end{bmatrix}.$$

Using

$$\beta^* f (f^* f)^k e_{m,n} = \delta_{n,0} 2^{2k+1} \, |q|^{2km+2m} \, e_{m,0}$$

$$-\beta f^* (f f^*)^k e_{m,n} = -\delta_{n,1} 2^{2k+1} |q|^{2km+2m} \, e_{m,1}$$

it follows that

$$\mathrm{Tr}(v^\dagger - 1)[\tilde{F}, v]([\tilde{F}, v^\dagger][\tilde{F}, v])^k = \sum_m^\infty (-)^k 2^{2k+1} |q|^{2km+2m} [1 - |q|^{2k+2}] = (-)^k 2^{2k+1}$$

Hence from (.19) $< [v], \mathbf{z} >= 1$, as stated.

Let now $B = C(S^4_q) := \Sigma C(S^3_q)$, see also [Pfl], [DLM]. [DL].
Using in Theorem .1 the n-th power $v^n \in M_2(C(S^3_q))$ of v, $n \in \mathbb{Z}$, we obtain the explicit projector $p_n \in M_4(C(S^4_q))$.
Its pairing (.5) with a preimage $\mathbf{w} \in K^0(C(S^4_q))$ of $\mathbf{z}$ under the surjective map δ is

$$< [p_n], \mathbf{w} >= n \ . \tag{.21}$$

This justifies the terminology (quantum) "instanton projection of charge n".

We mention that a q-deformation of $HH^\dagger$ given by (.13) (with charge 1) has been constructed in [DLM] and [DL], and with arbitrary integer charge in [L] (without computing their pairing with K-homology). Since the formulae (.11),(.12) for the change of coordinates are well defined in this noncommutative situation (the variables τ and t being central) we have also the (q-deformed) equivalent projectors $GG^\dagger$ and $ZZ^\dagger$ which hence have the same pairing (as well as their higher charge analogues).

The θ-deformed S^3 case

Next pass to the case $q \in \mathbb{C}$ with $|q| = 1$ and denote $q = \lambda = e^{i\theta}$, $\theta \in \mathbb{R}$, and $A = C(S^3_\theta)$, describing virtual θ-deformed 3-sphere.
Actually, I can and will work with ∞-parameter deformation $A = C(S^3_\Theta)$, described by $\Theta \in C([0,1], \mathbb{R})$ [Mat]. Just set in the commutation rules $q = \lambda = e^{i\Theta(\beta^*\beta)}$ (in the sense of continuous functional calculus).

Both K-groups of $C(S^3_\Theta)$ are isomorphic to $\mathbb{Z}$ and the generator of K_0 is given by [1] (so our assumptions are satisfied). Consider the matrix

$$v = \begin{bmatrix} \alpha & -\bar{\lambda}\beta^* \\ \beta & \alpha^* \end{bmatrix} \in M_2(C(S^3_\Theta)), \qquad (.22)$$

which is invertible, and in fact unitary. For $\Theta = \theta \in \mathbb{R}$, its pairing with certain spectral triple (unbounded Fredholm module) on S^3_θ was calculated in [L] to be 1 by reducing to the index pairing of the classical Dirac operator with the matrix (.6).
Preliminary computations indicate that this is the case also for S^3_Θ.

Let now $B = C(S^4_\Theta) := \Sigma C(S^3_\Theta)$. For $\Theta = \theta \in \mathbb{R}$, in [CL] a projection $p_\theta \in M_4(C(S^4_\theta))$ was introduced, (a θ-deformation of $HH^\dagger$ given by (.13)), and the pairing of p_θ with certain spectral triple on S^4_θ was shown to be 1.
This was achieved by applying the local index theorem of Connes and Moscovici, which simplifies considerably provided only the top component of the Chern character does not vanish. That this holds for S^4_θ was shown by a direct computation, containing few hundreds of terms. Next by another long computation the partial (matrix) trace of the local index formula was shown to be equal to the chirality operator γ_5. The task was then completed due to the fact that the remaining noncommutative integral to be performed involves the Dirac operator on S^4_θ which is isospectral to the classical one.

Similar computations for certain θ-deformed projectors $p_{\theta n} \in M_4(C(S^4_\theta))$ yield the value n for the pairing with v [L], which can thus be called "instanton projections of charge n". All these computations acquire a particularly simple explanation in view of the compatibility of the pairing of $K_\bullet$ with $K^\bullet$ and the equivalence (homotopy) between $p_{\theta n}$ and the projector associated by Theorem .1 to the n-th power $v^n \in M_2(C(S^3_\theta))$ of v given by (.22). Moreover, this extends to the new case of $C(S^3_\Theta)$.

Final comments

The methods presented above apply to a more general situation of pullback (fiber product).

The related problem of constructing the noncommutative analogue of associated principal fibre bundles, will be treated elsewhere [DHH].

Our results of computations of pairings of K-groups of A and ΣA should be further elucidated by finding a natural relation between the Fredholm modules and between the spectral triples on these algebras (smooth structure, metric and Dirac operator).

It should be mentioned that to get actually the "noncommutative instanton", one should next find a noncommutative analogue of the Yang-Mills action with critical points given by connections with (anti) selfdual curvature.
For other approches to noncommutative instanton vector bundles see [LM, HL, BCT, LPR].
Moreover, different types of non isomorphic C*-algebras of quantum spheres appeared e.g. in [CD-V], but their K-goups have not been computed yet.

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