θ-deformed Quantum Fields on the Noncommutative Minkowski Space

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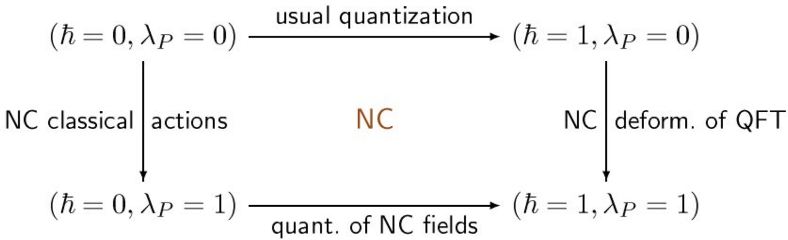
Classical gravity and quantum uncertainty → non-commutative spacetime.

- "two quantizations":
  $\hbar$ measures quantum nature of matter
  λ_P measures quantum nature of geometry



- Different (inequivalent!) constructions possible: ↓→ or →↓.
- Here: "→↓", i.e. investigate NC effects on quantum field theory

Quantum Fields on Non-Commutative Minkowski Space

Ingredients for the construction:

1. (Simple) model for NC Minkowski space: Selfadjoint coordinate operators $X_0, ..., X_3$ satisfying

$$[X_\mu, X_\nu] = i\,\theta_{\mu\nu} \cdot 1\,,$$

regularly represented on some Hilbert space $\mathcal{V}$, e.g. $\mathcal{V} = L^2(\mathbb{R}^2)$.

2. Description of undeformed QFT: Wightman framework
   - ϕ: (scalar) quantum field on commutative Minkowski space $\mathbb{R}^4$,
   - formulated as operator-valued distribution on Hilbert space $\mathcal{H}$
   - On $\mathcal{H}$: Unitary positive energy representation U of Poincare group, with vacuum vector $\Omega \in \mathcal{H}$
   - Usual locality and covariance requirements:

$$[\phi(x), \phi(y)] = 0 \qquad (x-y)^2 < 0$$
$$U(y,\Lambda)\phi(x)U(y,\Lambda)^{-1} = \phi(\Lambda x + y)$$

- On the X_μ, the translations act via

$$X_\mu \longmapsto X_\mu + x_\mu \cdot 1, \qquad x \in \mathbb{R}^4 .$$

- Suggestion for deformed field operator on NC $\mathbb{R}^4$ ([DFR] for free case)

$$\phi^\otimes(x) := \int d^4p\, e^{ip\cdot(X+x)} \otimes \tilde{\phi}(p)$$

- $\phi^\otimes$ can be rigorously defined as operator-valued distribution on dense domain in $\mathcal{V} \otimes \mathcal{H}$
- Polynomial algebra of the smeared fields $\phi^\otimes(f)$ replaces the field algebra of the QFT on commutative $\mathbb{R}^4$

- For doing QFT, need also a vacuum state
- simplest suggestion for vacuum state: product states

$$\omega := \nu \otimes \langle \Omega, \,.\, \Omega \rangle,$$

  with some states ν on the algebra of the X_μ
- "no correlations between field and geometry degrees of freedom"

The vacuum representation

- The state $\omega = \nu \otimes \langle \Omega, \, . \, \Omega \rangle$ is actually independent of ν:

$$\omega(\phi^{\otimes}(x_1) \cdots \phi^{\otimes}(x_n))$$
$$= \int dp_1 \cdots dp_n \, \nu(e^{iX \cdot \sum_{k=1}^n p_k}) \prod_{1 \leq l < r \leq n} e^{-\frac{i}{2} p_l \theta p_r} \langle \Omega, \tilde{\phi}(p_1) \cdots \tilde{\phi}(p_n) \Omega \rangle$$
$$= \int dp_1 \cdots dp_n \prod_{1 \leq l < r \leq n} e^{-\frac{i}{2} p_l \theta p_r} \langle \Omega, \tilde{\phi}(p_1) \cdots \tilde{\phi}(p_n) \Omega \rangle$$

  because of translation invariance of Ω
- Same procedure can be used for any translationally invariant state, e.g. thermal equilibrium states
- Given algebra of fields $\phi^{\otimes}(x)$ and state ω, go over to vacuum (GNS) representation

- GNS construction yields $(\mathcal{H}_\omega, \Omega_\omega, \pi_\omega)$:

$$\mathcal{H}_\omega = \mathcal{H}$$
$$\Omega_\omega = \Omega$$
$$\tilde{\phi}^\theta(p) := \pi_\omega(\tilde{\phi}^\otimes(p)) = \tilde{\phi}(p)\, e^{-\frac{i}{2}p\theta P},$$

with $U(y,1) = e^{iy_\mu P^\mu}$ energy-momentum operators of undeformed theory.

- Rigorous definition with twisted tensor product on algebra of test functions
- Example: Free scalar massive field. Here $\phi^\otimes$ is made out of annihilation/creation operators (on $\mathcal{V} \otimes \mathcal{H}$)

$$a_\otimes(p)^* := e^{ip\cdot X} \otimes a(p)^*, \qquad a_\otimes(p) := e^{-ip\cdot X} \otimes a(p),$$

and the GNS-represented field out of

$$a(\theta,p)^* := e^{-\frac{i}{2}p\theta P} a(p)^*, \qquad a(\theta,p)^* \qquad := e^{\frac{i}{2}p\theta P} a(p)^*$$

(on $\mathcal{H}$). [Akofor/Balachandran/Jo/Joseph 07, Grosse 79, GL 06, ...]

- n-point functions of the deformed fields:

$$\langle \Omega, \tilde{\phi}^\theta(p_1) \cdots \tilde{\phi}^\theta(p_n)\Omega\rangle = \prod_{1 \le l, r \le n} e^{-\frac{i}{2} p_l \theta p_r} \cdot \langle \Omega, \tilde{\phi}(p_1) \cdots \tilde{\phi}(p_n)\Omega\rangle$$

- continuous commutative limit (in n-point functions)
- The deformation $\phi \to \phi^\theta$ can also be defined in a more general operator-algebraic setting [Buchholz/Summers]

- **Here:** Stick to the field-theoretic setting, and study properties of ϕ^θ.
- In particular: ϕ^θ is neither local nor covariant if $\theta \neq 0$,

$$[\phi^\theta(x), \phi^\theta(y)] \neq 0 \qquad (x-y)^2 < 0$$
$$U(y,\Lambda)\phi^\theta(x)U(y,\Lambda)^{-1} \neq \phi^\theta(\Lambda x + y)$$

Covariance Properties of ϕ^θ

- Consider usual "untwisted" representation U of Poincaré group on $\mathcal{H}$: ($(y,\Lambda)x = \Lambda x + y$, $j(x) = -x$ total reflection)
- Transformation behaviour of $\phi^\theta(x)$ under U can be computed:

$$U(y,\Lambda)\phi^\theta(x)U(y,\Lambda)^{-1} = \phi^{\pm\Lambda\theta\Lambda^T}(\Lambda x + y)\,.$$

- $\Lambda\theta\Lambda^T = \theta$ for all Lorentz transformations Λ only possible for $\theta = 0$
- $\Rightarrow \phi^\theta(x)$ is not covariant for fixed $\theta \neq 0$.
- Lorentz symmetry generates family of fields

$$\{\phi^\theta \,:\, \theta \in \Theta\}$$

with Lorentz orbit $\Theta = \{\Lambda\theta_1\Lambda^T \,:\, \Lambda \in \mathcal{L}\}$ and reference noncommutativity θ_1

Covariance Properties of $\phi^\theta(x)$

- Transformation behaviour $\phi^\theta(x) \to \phi^{\Lambda\theta\Lambda^T}(\Lambda x + y)$ similar to string-localized fields [Mund/Schroer/Yngvason 05]
- $\to$ does $\phi^\theta(x)$ describe an extended field configuration?
- For the "standard θ" in $d = 4$ dimensions,

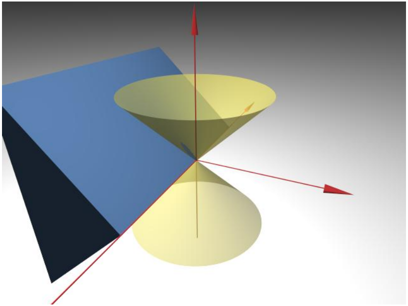
$$\theta = \theta_1 = \vartheta \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \vartheta \neq 0,$$

  we have $\Lambda\theta_1\Lambda^T = \theta_1$ only for
  - $\Lambda =$ Boost in x_1-direction
  - $\Lambda =$ Rotation in x_2-x_3-plane
- These are precisely the symmetries of the wedge region

$$W_1 = \{x \in \mathbb{R}^4 \,:\, x_1 > |x_0|\}$$

Wedges in $\mathbb{R}^d$

- Reference region $W_1 := \{x \in \mathbb{R}^d \,:\, x_1 > |x_0|\}$
- Set of wedges: $\mathcal{W}_0 := \mathcal{L}W_1$ (Lorentz transforms of W_1)
- $W \in \mathcal{W}_0$ satisfies $W' = -W$.
- Pictures in $d = 3$:



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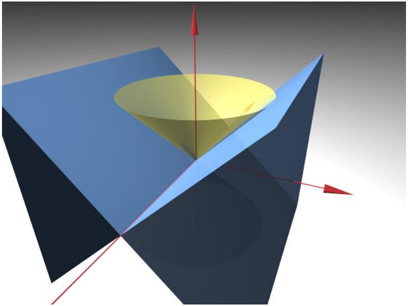

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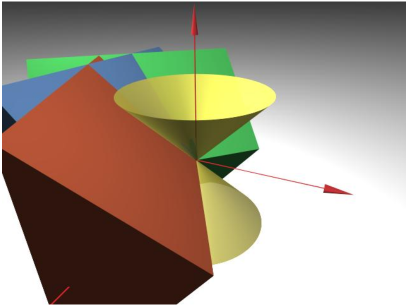

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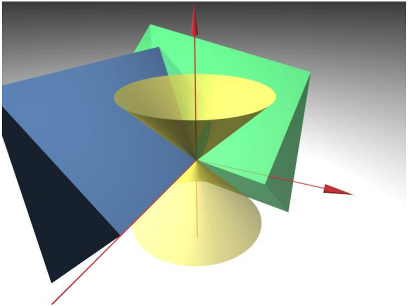


- As homogeneous spaces for the proper Lorentz group, $\mathcal{W}_0$ and Θ are isomorphic:

$$\theta : \mathcal{W}_0 \longrightarrow \Theta, \qquad \theta(\Lambda W_1) := \pm \Lambda \theta_1 \Lambda^T$$

- $\Rightarrow$ noncommutativity corresponding to causal complement:

$$\theta(W') = \theta(-W) = -\theta(W), \quad W \in \mathcal{W}_0 \, .$$

- P, T broken in $d = 4$, but TCP not, i.e. $j : x \mapsto -x$ is a symmetry

- Matching of symmetries of wedges and nc. parameters
- As far as covariance is concerned, $\phi^\theta(x)$ can consistently be interpreted as being localized in the wedge region $W(\theta) + x$.

- Is $\phi^\theta(x)$ localized in $W(\theta) + x$ in the sense of Einstein, i.e.

$$[\phi^\theta(x), \phi^{\theta'}(x')] = 0 \quad \text{for} \quad (W(\theta) + x) \subset (W(\theta') + x')' \ \textbf{?}$$

- The condition that $W(\theta) + x$ and $W(\theta') + x'$ are spacelike separated is strong: It implies in particular $\theta' = -\theta$.
- sufficient to consider $[\phi^\theta(x), \phi^{-\theta}(x')]$ with $x \in W(\theta)$, $x' \in -W(\theta)$.
- In the example of the deformed free field, consider full algebra of creation/ann. operators:

$$a(\theta,p)a(\theta',p') = e^{-\frac{i}{2}p(\theta+\theta')p'}a(\theta',p')a(\theta,p)$$

$$a^*(\theta,p)a^*(\theta',p') = e^{-\frac{i}{2}p(\theta+\theta')p'}a^*(\theta',p')a^*(\theta,p)$$

$$a(\theta,p)a^*(\theta',p') = e^{+\frac{i}{2}p(\theta+\theta')p'}a^*(\theta',p')a(\theta,p) + \omega_{\boldsymbol{p}}\delta(\boldsymbol{p}-\boldsymbol{p}')e^{\frac{i}{2}p(\theta-\theta')P}$$

- many cancellations for $\theta' = -\theta \Rightarrow$ deformed free field is wedge-local.

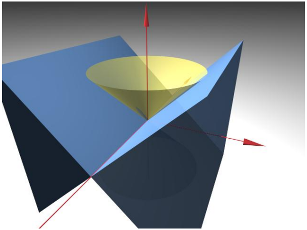
Situation for general quantum fields:

- If the undeformed field ϕ is local and the energy is positive in every Lorentz frame (spectrum condition), then the deformed field operator $\phi^\theta(x)$ is localized in the wedge $W(\theta) + x$.
- First proof in operator-algebraic setting by [Buchholz-Summers 08], then in a field-theoretic setting [Grosse-GL 08].

- Wedge-locality is a remnant of the usual locality which is compatible with noncommutativity

Scattering processes

- Observable consequences of the deformation? Investigate
  - Scattering processes (here)
  - Also interesting: Thermal correlations [Grosse/GL, work in progress]
- In scattering theory, need to separate single particle states asymptotically $\rightarrow$ Non-locality of $\phi^\theta(x)$ problematic
- but wedge-locality allows causal separation of two wedges



- $\Rightarrow$ two-particle scattering can be done (Method: Haag-Ruelle scattering theory)
- Construct two-particle states with the right asymptotic localization and momentum space properties [Borchers/Buchholz/Schroer 00]

- Two-particle scattering states can be computed
- they depend on non-commutativity (choice of wedge-fields)

$$_{\rm in}\langle p, \tilde{p} | q, \tilde{q} \rangle_{\rm out} = e^{ip\theta_1 \tilde{p}} \langle p, \tilde{p} | q, \tilde{q} \rangle \qquad \text{for} \quad p_1 > \tilde{p}_1, \; q_1 > \tilde{q}_1$$

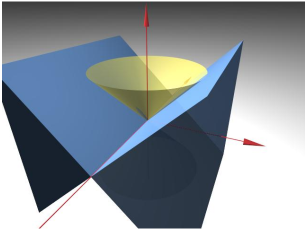
- NC leads to change of S-matrix: non-trivial scattering!
- despite the Lorentz covariance of the model, the S-matrix breaks the Lorentz symmetry
- similar to "background field"
- $|e^{ip\theta q}| = 1 \Rightarrow$ No change in cross sections, but in time delays
- Situation similar to integrable models in $d = 1 + 1$

Optimal localization

- The noncommutativity

$$\theta = \vartheta \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

  has two commuting ("classical") directions
- $\rightarrow$ Sharp localization in two directions should be possible, i.e. in intersection of two opposite wedges



- Do there exist such "optimally localized" observables in our model?

- An optimally localized observable A must satisfy

$$[A, \phi_1^\theta(x)] = [A, \phi^{-\theta_1}(x')] = 0 \qquad x_1 > \varepsilon,\ x_1' < -\varepsilon$$

- Set $\mathcal{A}_\varepsilon$ of all solutions of this condition is a (v. Neumann) algebra.
- $\mathcal{A}_\varepsilon \neq \mathbb{C} \cdot 1$? question still open

Same method can be applied to find algebras $\mathcal{A}(\mathcal{O})$ of observables localized in bounded spacetime regions $\mathcal{O} \subset \mathbb{R}^4$.

- $\overline{\mathcal{A}(\mathcal{O})\Omega} \neq \mathcal{H}$ local violation of Reeh-Schlieder property
- Model defined by the fields ϕ^θ is not generated by a local QFT ("intrinsic nonlocality")
- Probably even $\mathcal{A}(\mathcal{O}) = \mathbb{C} \cdot 1$ (no local observables at all)

Conclusions and open questions

New family of model QFTs:

- deformation of fields on comm. Minkowski space to fields on NC Minkowski space
- Example: related to "free" field on NC Minkowski space
- Consequent application of Poincaré symmetry leads to wedge-local fields
- Remnants of Covariance and Locality found in NC model:

$$U(y,\Lambda)\phi^\theta(x)U(y,\Lambda)^{-1} = \phi^{\pm\Lambda\theta\Lambda^T}(\Lambda x + y)\,.$$

$$(W(\theta)+x) \subset (W(\theta')+x')' \Longrightarrow [\phi^\theta(x), \phi^{\theta'}(x')] = 0\,.$$

- Two-particle scattering can be computed, and S-Matrix becomes non-trivial

Properties of the NC deformation

- local fields $\longrightarrow$ wedge-local fields
- free fields $\longrightarrow$ interacting fields

- Comparison to usual approach starting from $\star_\theta$-deformed action? (Phases on Feynman diagrams differ)
- Euclidean formulation also possible.
  Passage Euclidean $\leftrightarrow$ Minkowskian in this setting probably manageable [Grosse/GL, work in progress]