Noncommutative Black Holes & Quantum Structure of Space-time

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Plan of the Talk:

- Motivation
- Noncommutative geometry
- Diffeomorphisms Symmetry in the NC framework
- NC Black Holes
- QFT in the NC Black Hole background
- Concluding remarks

Motivation

- Two basic questions about nature are:
  - Can events in space-time be localized with arbitrary precision ?
  - Is there a fundamental and elementary length scale in nature ?
- These issues are related to the space-time structure at the Planck scale.
- Noncommutative Geometry is one of the candidates for describing physics at that regime.

Space-time UR

Heisenberg's Principle
+ $\Longrightarrow$ Space-time uncertainty relations
Einstein's Theory

- A measurement involving space-time coordinates with an accuracy δ causes and uncertainty in the momentum $\sim \frac{1}{\delta}$.
- In this process, an energy of the order $\frac{1}{\delta}$ is transmitted to the system and concentrated for some time in the localization region. The associated energy-momentum tensor generates a gravitational field.
- The smaller the uncertainties in the measurement of coordinates, the stronger will be the gravitational field generated by the measurement.

Space-time UR

- To probe physics at Planck Scale l_p, the Compton wavelength $\frac{1}{M}$ of the probe must be less than l_p, hence $M > \frac{1}{l_p}$, i.e. Planck mass.
- When this field becomes so strong as to prevent light or other signals from leaving the region in question, the concept of localization becomes fuzzy.
- Similarly, observations of very short time scales also require very high energies. Such observations can also form black holes and limit spatial resolutions leading to a relation of the form

$$\Delta t \Delta x \geq L^2, \quad L = \text{fundamental length}$$

.

Space-time UR

- Based on similar arguments, Doplicher, Fredenhagen and Roberts (1994) arrived at uncertainty relations between the coordinates, which they showed could be deduced from a commutation relation of the type

$$[q_\mu, q_\nu] = iQ_{\mu\nu}$$

where q_μ are self-adjoint coordinate operators, μ, ν run over space-time coordinates and $Q_{\mu\nu}$ is an antisymmetric tensor, with the simplest possibility that it commutes with the coordinate operators.

Noncommutative geometry

An example of noncommutative geometry is provided by the d-dimensional Groenewold-Moyal spacetime or GM plane, which is an algebra $\mathcal{A}_\theta(\mathbb{R}^N)$ generated by elements $\hat{x}_\mu$ ($\mu \in [0, 1, 2, \cdots, N-1]$) with the commutation relation

$$[\hat{x}_\mu\,, \hat{x}_\nu] = i\theta_{\mu\nu}\mathbf{1}\,,$$

$\theta_{\mu\nu}$ being real, constant and antisymmetric in its indices. This algebra can be represented by functions of commuting variables with a twisted product

$$f * g = f e^{i/2 \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu} g = F_\theta f g.$$

The $*$ product defines the associative but noncommutative algebra $\mathcal{A}_\theta(\mathbb{R}^N)$. The twist element is denoted by F_θ.

NCG

- In the commutative case, we know that general relativity is a theory invariant under the symmetry group of diffeomorphisms. Assuming this continues to hold at the Planck scale where noncommutative geometry is supposed to be relevant, we must address the issue of how diffeomorphism symmetry acts on the algebra $\mathcal{A}_\theta(\mathbb{R}^N)$.
- For that, we first discuss how a symmetry group acts on a general algebra.

Symmetry on algebra

- Let $\mathcal{A}$ be an algebra. $\mathcal{A}$ comes with a rule for multiplying its elements. For $f, g \in \mathcal{A}$ there exists the multiplication map μ such that

$$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \,,$$
$$f \otimes g \to \mu(f \otimes g) \,.$$

- Now let $\mathcal{G}$ be the group of symmetries acting on $\mathcal{A}$ by a given representation $D : \alpha \to D(\alpha)$ for $\alpha \in \mathcal{G}$. We can denote this action by

$$f \longrightarrow D(\alpha) f \,.$$

Symmetry on algebra

The action of $\mathcal{G}$ on $\mathcal{A} \otimes \mathcal{A}$ is formally implemented by the coproduct Δ

$$\Delta : \mathcal{G} \longrightarrow \mathcal{G} \otimes \mathcal{G}$$

The action is compatible with μ only if a certain compatibility condition between $\Delta(\alpha)$ and μ is satisfied. This action is

$$f \otimes g \longrightarrow (D \otimes D)\Delta(\alpha) f \otimes g \,,$$

and the compatibility condition requires that

$$\mu \left((D \otimes D)\Delta(\alpha) f \otimes g \right) = D(\alpha)\, \mu(f \otimes g) \,.$$

Symmetry on algebra

The compatibility condition can be expressed in terms of the following commutative diagram :

$$
\begin{array}{ccc}
f \otimes g & \xrightarrow{\quad \Delta \quad} & (D \otimes D)\Delta(\alpha) f \otimes g \\
\Big\downarrow \mu & & \Big\downarrow \mu \\
\mu(f \otimes g) & \xrightarrow{\qquad\qquad} & D(\alpha)\mu(f \otimes g)
\end{array}
$$

If a Δ satisfying the above compatibility condition exists, then $\mathcal{G}$ is an automorphism of $\mathcal{A}$. If such a Δ cannot be found, then $\mathcal{G}$ does not act on $\mathcal{A}$.

Commutative Diffeos

- Diffeos are generated by vector fields defined by

$$\xi = \xi^\mu \frac{\partial}{\partial x^\mu}$$

- Denote the space of vector fields by V. Commutator of two vector fields $\xi, \eta \in V$ is another vector field in V given by by

$$[\xi, \eta] = (\eta^\mu(\partial_\mu \xi^\rho) - \xi^\mu(\partial_\mu \eta^\rho)) \frac{\partial}{\partial x^\rho}$$

Leibnitz Rule

- The Leibniz rule for the diffeos is given by

$$(\xi(f.g)) = (\xi f).g + f.(\xi g)$$

  where $f, g \in \mathcal{A}_0(\mathbb{R}^N)$ and are multiplied by the usual commutative pointwise multiplication rule.

- Leibniz rule is equivalent to the coproduct for the diffeos

$$\Delta_0 : V \longrightarrow V \otimes V, \qquad \Delta_0(\xi) = \xi \otimes \mathbf{1} + \mathbf{1} \otimes \xi$$

- This coproduct or the Leibnitz rule is compatible with the multiplication map on the algebra of vector fields

$$[\Delta_0(\xi), \Delta_0(\eta)] = \Delta_0([\xi, \eta])$$

NC Diffeos

In the noncommutative case, we have the algebra $\mathcal{A}_\theta(\mathbb{R}^N)$ with the multiplication map μ_θ. Various works, based mainly on ideas of Drinfeld have shown that

- The coproduct Δ_0 is not compatible with the multiplication map μ_θ.
- One can define a new twisted coproduct

$$\Delta_\theta = F_\theta^{-1} \Delta_0 F_\theta$$

  which is compatible with μ_θ.
- This implies that the Leibniz rule is modified when $\theta \neq 0$.

Remarks

- The diffeomorphism symmetry can be implemented in the presence of space-time noncommutativity, one possible way is by using twisted coproducts.
- A formulation of NC GR has been proposed by Wess at al and NC analogue of Einstein's equations have been obtained.
- However, the solutions of the NC gravity equations are very hard to find. Only a few exact solutions in lower dimensions are known.

Remarks

- In our approach, we start with a commutative solution of Einstein's equations, and then find the Poisson brackets of the space-time variables which are consistent with the geometry of the solution.
- The noncommutative gravity solutions is then obtained by representation of the algebra as operators, or by "quantization".
- We shall do this for BTZ black holes in 2+1 dimensions.

BTZ

- The metric for the BTZ black hole in terms of Schwarzschild-like coordinates (r, t, ϕ) is given by

$$ds^2 = \left(M - \frac{r^2}{\ell^2} - \frac{J^2}{4r^2}\right)dt^2 + \left(-M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}\right)^{-1} dr^2 + r^2\left(d\phi - \frac{J}{2r^2}dt\right)^2 ,$$

$$0 \leq r < \infty \, , \; -\infty < t < \infty \, , \; 0 \leq \phi < 2\pi \, ,$$

where M and J are the mass and spin, respectively, and $\Lambda = -1/\ell^2$ is the cosmological constant.

- For $0 < |J| < M\ell$, there are two horizons, the outer and inner horizons, corresponding respectively to $r = r_+$ and $r = r_-$, where

$$r_\pm^2 = \frac{M\ell^2}{2}\left\{1 \pm \left[1 - \left(\frac{J}{M\ell}\right)^2\right]^{\frac{1}{2}}\right\}$$

- The metric is diagonal in the coordinates (χ_+, χ_-, r), where

$$\chi_\pm = \frac{r_\pm}{\ell}t - r_\mp\phi \, ,$$

BTZ

- The manifold of the BTZ black hole solution is the quotient space of the universal covering space of AdS^3 by some elements of the group of isometries of AdS^3.
- Let AdS^3 be spanned by coordinates (t_1, t_2, x_1, x_2) satisfying

$$-t_1^2 - t_2^2 + x_1^2 + x_2^2 = -\ell^2$$

- Alternatively, one can introduce 2×2 real matrices

$$g = \frac{1}{\ell} \begin{pmatrix} t_1 + x_1 & t_2 + x_2 \\ -t_2 + x_2 & t_1 - x_1 \end{pmatrix} \qquad \det g = 1 \ ,$$

belonging to the defining representation of $SL(2, R)$.

BTZ

- The isometries correspond to the left and right actions on g,

$$g \longrightarrow h_L g h_R \ , \qquad h_L, h_R \in SL(2,R)$$

- Since (h_L, h_R) and $(-h_L, -h_R)$ give the same action, the connected component of the isometry group for AdS^3 is

$$SL(2,R) \times SL(2,R)/Z_2 \approx SO(2,2)$$

.

BTZ

The BTZ black-hole is obtained by discrete identification of points on the universal covering space of AdS_3. This ensures periodicity in ϕ, $\phi \sim \phi + 2\pi$. The condition is

$$g \sim \tilde{h}_L g \tilde{h}_R \ , \qquad \tilde{h}_L, \tilde{h}_R \in SO(2,2)$$

where

$$\tilde{h}_L = \begin{pmatrix} e^{\pi(r_+ - r_-)/\ell} & 0 \\ 0 & e^{-\pi(r_+ - r_-)/\ell} \end{pmatrix}, \qquad \tilde{h}_R = \begin{pmatrix} e^{\pi(r_+ + r_-)/\ell} & 0 \\ 0 & e^{-\pi(r_+ + r_-)/\ell} \end{pmatrix}$$

Thus,

$$\mathrm{BTZ} = \frac{AdS^3}{< (\tilde{h}_L, \tilde{h}_R) >}$$

where $< (\tilde{h}_L, \tilde{h}_R) >$ denotes the group generated by $(\tilde{h}_L, \tilde{h}_R)$.

BTZ

- The identification breaks the $SO(2,2)$ group of isometries to a two-dimensional subgroup $\mathcal{G}_{BTZ}$, consisting of only the diagonal matrices in $\{h_L\}$ and $\{h_R\}$.
- $\mathcal{G}_{BTZ}$ is the isometry group of the BTZ black hole.

We shall now discuss the deformation of this solution.

NC BTZ

For generic spin, $0 < |J| < M\ell$ (and $M > 0$), we shall search for Poisson brackets for the matrix elements of g which are polynomial of lowest order. They should be consistent with the quotienting, as well as the unimodularity condition and the Jacobi identity.
Writing the $SL(2, R)$ matrix as

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \alpha\delta - \beta\gamma = 1 \ ,$$

Under the quotienting, we get

$$\begin{array}{ccc} \alpha & \sim & e^{\ 2\pi r_+/\ell} \alpha \\ \beta & \sim & e^{-2\pi r_-/\ell} \beta \\ \gamma & \sim & e^{\ 2\pi r_-/\ell} \gamma \\ \delta & \sim & e^{-2\pi r_+/\ell} \delta \end{array}$$

All quadratic combinations of matrix elements scale differently, except for $\alpha\delta$ and $\beta\gamma$, which are invariant under the quotienting.

NC BTZ

Lowest order polynomial expressions for the Poisson brackets of α, β, γ and δ which are preserved under the quotienting are quadratic and have the form

$$\{\alpha, \beta\} = c_1 \alpha\beta \;\; \{\alpha, \gamma\} = c_2 \alpha\gamma \;\; \{\alpha, \delta\} = f_1(\alpha\delta, \beta\gamma)$$

$$\{\beta, \delta\} = c_3 \beta\delta \;\; \{\gamma, \delta\} = c_4 \gamma\delta \;\; \{\beta, \gamma\} = f_2(\alpha\delta, \beta\gamma)$$

where c_{1-4} are constants and $f_{1,2}$ are functions.
They are constrained by

$$\begin{aligned} c_1 + c_2 &= c_3 + c_4 \\ f_1(\alpha\delta, \beta\gamma) &= (c_1 + c_2)\beta\gamma \\ f_2(\alpha\delta, \beta\gamma) &= (c_2 - c_4)\alpha\delta \; , \end{aligned}$$

after demanding that det$g$ is a Casimir of the algebra. There are three independent constants c_{1-4}.

NC BTZ

Further restrictions on the constants come from the Jacobi identity, which leads to the following two possibilities:

$$\text{A.} \quad c_2 = c_4 \qquad \text{and} \qquad \text{B.} \quad c_2 = -c_1$$

Both cases define two-parameter families of Poisson brackets. Say we call c_2 and c_3 the two independent parameters. The two cases are connected by an $SO(2,2)$ transformation.

NC BTZ

We can write the Poisson brackets for the various cases in terms of the Schwarzschild-like coordinates (r, t, ϕ). For the two-parameter families A and B we get

A.

$$
\begin{aligned}
\{\phi, t\} &= \frac{\ell^3}{2} \frac{c_3 - c_2}{r_+^2 - r_-^2} \\
\{r, \phi\} &= -\frac{\ell r_+ (c_3 + c_2)}{2r} \frac{r^2 - r_+^2}{r_+^2 - r_-^2} \\
\{r, t\} &= -\frac{\ell^2 r_- (c_3 + c_2)}{2r} \frac{r^2 - r_+^2}{r_+^2 - r_-^2}
\end{aligned}
$$

B.

$$
\begin{aligned}
\{\phi, t\} &= \frac{\ell^3}{2} \frac{c_3 - c_2}{r_+^2 - r_-^2} \\
\{r, \phi\} &= -\frac{\ell r_- (c_2 + c_3)}{2r} \frac{r^2 - r_-^2}{r_+^2 - r_-^2} \\
\{r, t\} &= -\frac{\ell^2 r_+ (c_2 + c_3)}{2r} \frac{r^2 - r_-^2}{r_+^2 - r_-^2}
\end{aligned}
$$

NC BTZ

These Poisson brackets are invariant under the action of the isometry group $\mathcal{G}_{BTZ}$ of the BTZ black hole. A central element of the Poisson algebra can be constructed out of the Schwarzschild coordinates for both cases. It is given by

$$\rho_\pm = (r^2 - r_\pm^2)\exp\left\{-\frac{2\kappa\chi_\pm}{\ell}\right\}, \qquad c_2 \neq c_3 \ ,$$

where the upper and lower sign correspond to case A and B, respectively,

$$\kappa = \frac{c_3 + c_2}{c_3 - c_2} \ ,$$

The $\rho_\pm =$constant surfaces define symplectic leaves, which are topologically $\mathbb{R}^2$ for generic values of the parameters (more specifically, $c_2 \neq \pm c_3$). We can coordinatize them by χ_+ and χ_-. One then has a trivial Poisson algebra in the coordinates $(\chi_+, \chi_-, \rho_\pm)$:

$$\{\chi_+, \chi_-\} = \frac{\ell^2}{2}(c_3 - c_2) \qquad \{\rho_\pm, \chi_+\} = \{\rho_\pm, \chi_-\} = 0$$

The action of the $\mathcal{G}_{BTZ}$ transforms one symplectic leaf to another, except for the case $c_2 = -c_3$, on which we focus from now on.

NC BTZ

- For $c_2 = -c_3$, the radial coordinate is in the center of the algebra.
- $r =$constant define $\mathbb{R} \times S^1$ symplectic leaves, and they are invariant under the action of $\mathcal{G}_{BTZ}$.
- The coordinates ϕ and t parametrizing any such surface are canonically conjugate:

$$\{\phi, t\} = \frac{c_3 \ell^3}{r_+^2 - r_-^2} \qquad \{\phi, r\} = \{t, r\} = 0$$

NC BTZ

- Upon passing to the "quantum" theory, in terms of the operators $\hat{\phi}$, $\hat{t}$ and $\hat{r}$, we have

$$[\hat{\phi}, \hat{t}] = i\theta \qquad [\hat{\phi}, \hat{r}] = [\hat{t}, \hat{r}] = 0$$

where the constant θ is linearly related to $\ell^3/(r_+^2 - r_-^2)$.

- Deformation of BTZ provides an example of the general space-time noncommutativity given by

$$[\hat{x}_0, \hat{x}_1] = i\theta$$

.

NC BTZ

- Since the coordinate ϕ is periodic, it is better to consider the operators $\hat{t}, e^{i\hat{\phi}}$ and $\hat{r}$, which satisfy :

$$[e^{i\hat{\phi}}, \hat{t}] = \theta e^{i\hat{\phi}} \qquad [\hat{r}, \hat{t}] = [\hat{r}, e^{i\hat{\phi}}] = 0 \ ,$$

- There are now two central elements in the algebra:

$$i) \ \hat{r} \qquad \text{and} \qquad ii) \ e^{-2\pi i\hat{t}/\theta}.$$

NC BTZ

- In an irreducible representation, the central element is proportional to the identity

$$e^{-2\pi i\hat{t}/\theta} = e^{i\chi}\mathbf{1}$$

- The spectrum of the time operator $\hat{t}$ is then discrete

$$n\theta - \frac{\chi\theta}{2\pi}\ , \quad n \in \mathbb{Z}$$

- If there is a Hamiltonian description for this analysis, then the corresponding energy is conserved modulo $\frac{2\pi}{\theta}$.

QFT in NC BH Background

- In standard QFT, the quantization of a field is done by mode expansion, imposition of suitable commutation relations on the creation and annihilation operators depending on the statistics of the field and finally obtaining a Fock space representation.
- In addition, both continuous and discrete symmetries must act properly on the fields.

Statistics

- Statistics of particles or quantum fields should be frame independent.
- In the commutative case, the Flip Operator τ_0, which acts on a 2-particle Hilbert space must satisfy

$$[\tau_0, \Delta_0] = 0.$$

- In the NC case, it turns out that

$$[\tau_0, \Delta_\theta] \neq 0.$$

- τ_0 can no longer be used to define particle/field statistics.

Twisted Statistics

- Can define a Twisted Flip Operator τ_θ given by

$$\tau_\theta = F^{-1} \tau_0 F$$

which satisfies

$$[\tau_\theta, \Delta_\theta] = 0$$

- Particle statistics will be governed by τ_θ.
- This will lead to a new algebra of the creation and annihilation operators for the quantum fields in NC BTZ space-time, which would have novel physical effects.

Discrete Symmetries

- Consider the relation

$$[x_0, x_1] = i\theta$$

- Under P, $x_1 \to -x_1$, x_0 and $i\theta$ unchanged. Hence P is not an automorphism.
- Under T, $x_0 \to -x_0$, $i\theta \to -i\theta$, x_1 unchanged. Hence T is an automorphism.
- As a result, if C is conserved, P, PT and CPT violated.
- Precision measurements can put bounds on the noncommutativity parameter.

Continuous Symmetries

- We have seen that the deformation quantization in the BTZ leads to discrete time and hence to discrete time translations.
- This means that the isometry of the classical system is broken due to the quantization.
- This effect can be thought of as a noncommutative anomaly and any QFT on such a background must take this effect into account.
- Construction of appropriate QFT is in progress.

Concluding remarks

- Noncommutative geometry provides a natural framework for quantum theory of gravity.
- Deformation of the BTZ black hole solutions in 2+1 dimensions which is consistent with the classical geometry leads to space-time noncommutativity.
- The most natural choice for the deformation of the BTZ leads to a quantum structure of space-time, with time and time translation quantized.

Concluding remarks

- QFT in such a space-time would be governed by twisted commutation relations of the creation and annihilation operators, with the possibility of novel physical effects.
- Such a model would lead to violation of P, PT and CPT, leading to empirical bounds on the noncommutativity parameter.
- Commutative BTZ is closely related to holography. With NC BTZ, the idea of noncommutative holography can be explored.