

DIMENSIONAL REDUCTION, MONOPOLES

AND DYNAMICAL SYMMETRY BREAKING

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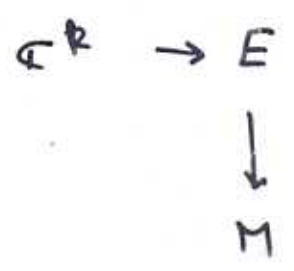
OUTLINE

- EQUIVARIANT DIMENSIONAL REDUCTION ($M \times S^2$)
 - GAUGE FIELDS $U(\mathbb{R}) \rightarrow U(\mathbb{R}_0) \times U(\mathbb{R}_1) \times \dots \times U(\mathbb{R}_m)$
 - SPINOR FIELDS AND YUKAWA COUPLINGS
 - DYNAMICAL SYMMETRY BREAKING
- FUZZY S^2
 - SPINOR FIELDS, UNIVERSAL DIRAC OPERATOR

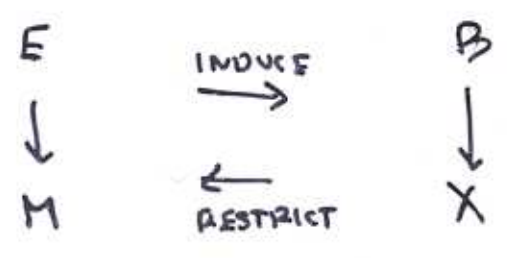
EQUIVARIANT DIMENSIONAL REDUCTION

$X = M \times S^2$: GAUGE GROUP $G = U(1)$
 \uparrow d DIMENSIONAL

$S^2 \approx SU(2)/U(1)$, $SU(2)$ ACTS TRIVALLY ON M



INDUCE A BUNDLE $B = SU(2) \times_{U(1)} E$ OVER X
 $g \in SU(2), e \in E, h \in U(1) : h(g, e) = (gh^{-1}, he)$



$B : SU(2)$ EQUIVARIANT BUNDLE OVER X
 $E : U(1)$ " " " M
 $\Rightarrow E = \bigoplus_P E(p)$ $p \in \mathbb{Z}$ LABELS IRREPS. OF $U(1)$
 i.e. CHARGE
 \mathbb{C}^R IS A REPRESENTATION SPACE FOR $SU(2)$ ($0, m, m+1$)

$$\mathbb{C}^R = \bigoplus_{i=0}^m \mathbb{C} A_i \qquad \sum_{i=0}^m A_i = m R$$

$$L_G = \begin{pmatrix} 5/1 & & & & \\ & 1 & & & \\ & & 5/1 & & \\ & & & \dots & \\ & & & & 5/1 \\ & & & & & \dots & \\ & & & & & & 5/1 \end{pmatrix} \qquad \begin{array}{l} 5/1 \in \mathbb{C}^R \\ 1 \in \mathbb{C}^R \\ 5/1 \in \mathbb{C}^R \\ \dots \\ 5/1 \in \mathbb{C}^R \end{array}$$

$$B = \bigoplus_{i=0}^3 B_i, \quad B_i = E(p_i) \times \mathcal{L}^{p_i}$$

LINE BUNDLE OVER S^2 ,
CHERN NUMBER p_i

$$\mathbb{C} \rightarrow \mathcal{L}^{p_i}$$

$$\downarrow$$

$$S^2$$

$$\mathbb{C}^{p_i} \rightarrow E(p_i) = E_i$$

$$\downarrow$$

$$M$$

SU(2) GENERATORS : $\tau_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\tau_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$[\tau_3, \tau_{\pm}] = \pm 2 \tau_{\pm}$$

$$\tau_{\pm} : E(p) \rightarrow E(p \pm 2)$$

$$\tau_{\pm} : E(p_i) \times \mathcal{L}^{p_i} \rightarrow E(p_i \pm 2) \times \mathcal{L}^{p_i \pm 2}$$

τ_{\pm} ACTION DETERMINED BY A CHAIN :

$$0 \rightarrow B_m \xrightarrow{\Phi_m} B_{m-1} \xrightarrow{\Phi_{m-1}} \dots \xrightarrow{\Phi_1} B_1 \xrightarrow{\Phi_0} B_0 \rightarrow 0$$

$$U(k_m) \quad U(k_{m-1}) \quad \quad \quad U(k_1) \quad U(k_0)$$

$$p_i = m - 2i$$

CURVATURE

x^{μ}, x^{ν} : CO-ORDINATES ON X , METRIC $g_{\mu\nu}$

x^{μ}, x^{ν} : CO-ORDINATES ON M , $g_{\mu\nu}$

y, \bar{y} : CO-ORDINATES ON S^2 , $g_{y\bar{y}} = \frac{4R^2}{(1+y\bar{y})^2}$

$\beta = \frac{2 dy}{(1+y\bar{y})}$, $\bar{\beta} = \frac{2 d\bar{y}}{(1+y\bar{y})}$: ONE-FORMS ON S^2

MONOPOLE POTENTIAL: $a_p = \frac{f}{2(1+y\bar{y})} (\bar{y} dy - y d\bar{y})$

$$\mathcal{F}_p = - \frac{f}{(1+y\bar{y})^2} dy \wedge d\bar{y}$$

$$\frac{i}{2\pi} \int_{S^2} \mathcal{F}_p = p \quad : \text{FIRST CHERN NUMBER FOR } \mathcal{L}^p$$

$U(\mathcal{R})$ GAUGE POTENTIAL ON X , $\mathcal{A} (= -\mathcal{A}^\dagger)$
SPLITS INTO $\mathcal{R}_i \times \mathcal{R}_i$ BLOCKS

$$\mathcal{A} = A(x) + a(y) + \Phi(x) \bar{\beta}(y) - \Phi^\dagger(x) \beta(y)$$

$$A = \sum_{i=0}^m A^i, \quad a = \sum_{i=0}^m a_{m-2i}$$

Φ \rightarrow $\mathcal{R} \times \mathcal{R}$ MATRIX

$$\Phi = \begin{pmatrix} 0 & \phi_1 & 0 & \dots & 0 \\ 0 & 0 & \phi_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \phi_m \end{pmatrix}$$

$$\phi_i: \mathcal{R}_{i-1} \times \mathcal{R}_i$$

(4)

$$A^{i,i} = A^i + \alpha_{m-2i} \quad \leftarrow U(A^i) \text{ GAUGE POTENTIAL}$$

$$A^{i,i+1} = \phi_{i+1}(x) \bar{\beta}(y)$$

FIELD STRENGTH ON X : $\mathcal{F} = dA + A \wedge A$

$$\mathcal{F} = F + \mathcal{F} + [\Phi, \Phi^+] \beta \wedge \bar{\beta} + D\Phi \wedge \bar{\beta} - D\Phi^+ \wedge \beta$$

$$D\Phi = d\Phi + [A, \Phi]$$

$$F = \sum_{i=0}^m F^i \quad \mathcal{F} = \sum_{i=0}^m \mathcal{F}_{m-2i}$$

$$\mathcal{F}_{\mu\nu}^{i,i} = F_{\mu\nu}^i(x)$$

$$\mathcal{F}_{y\bar{y}}^{i,i} = -\frac{1}{(1+y\bar{y})^2} (m-2i + 4\phi_i^\dagger \phi_i - 4\phi_{i+1} \phi_{i+1}^\dagger)$$

$$\mathcal{F}_{\mu\bar{y}}^{i,i+1} = \frac{2}{(1+y\bar{y})} D_\mu \phi_{i+1}(x) = -(\mathcal{F}_{\mu\bar{y}}^{i+1,i})^\dagger$$

$$D\phi_{i+1} = d\phi_{i+1} + A^i \phi_{i+1} - \phi_{i+1} A^{i+1}$$

ACTION ON X:

$$\hat{S} = -\frac{1}{4\tilde{g}^2} \int d^d x dy d\tilde{y} \sqrt{\tilde{g}} \text{tr}_R F_{\tilde{\mu}\tilde{\nu}} F^{\tilde{\mu}\tilde{\nu}}$$

$\tilde{g} : SU(R)$ GAUGE COUPLING (OVERALL $V(\Phi)$ DROPS OUT)

INTEGRATE OVER $S^2 \rightarrow$ ACTION ON M:

$$S = \int d^d x \sqrt{g} \left\{ -\frac{1}{g^2} F_{\mu\nu} F^{\mu\nu} + (D_\mu \Phi)^\dagger D_\mu \Phi + V(\Phi) \right\}$$

$$V(\Phi) = \frac{g^2}{2} \left(\frac{1}{4g^2 R^2} [\Phi, \Phi^\dagger]^2 \right)^2$$

$$\Sigma = \begin{pmatrix} m \mathbb{1}_{R_0} & & & \\ & (m-2) \mathbb{1}_{R_1} & & 0 \\ & & \ddots & \\ 0 & & & -m \mathbb{1}_{R_m} \end{pmatrix}$$

$g^2 = \frac{2g^2}{4\pi R^2}$
 \uparrow
d-DIMENSIONAL GAUGE COUPLING

$V(\Phi)$ CAN LEAD TO SYMMETRY BREAKING:

$$\frac{\delta V}{\delta \Phi} = 0 \Rightarrow [\Phi, [\Phi, \Phi^\dagger]] = \frac{1}{4g^2 R^2} [\Phi, \Sigma]$$

$$[\Phi, \Phi^\dagger] = \frac{i}{4g^2 R^2} \Sigma$$

e.g. $k_0 = k_1 = \dots = k_m = n$; $R = n(m+1)$

$$SU(R) \rightarrow SU(n)^{m+1} \times U(1)^m$$

$$\Phi^0 = \begin{pmatrix} 0 & \phi_1^0 & 0 & \dots & 0 \\ 0 & 0 & \phi_2^0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \phi_m^0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\phi_i^0 = \frac{\sqrt{i(m-i+1)}}{2gR} \mathbf{1}_n$$

BREAKS $SU(n)^{m+1} \times U(1)^m \rightarrow SU(n)$

$m=1$, $\phi_1 = \frac{1}{2gR} + h$, $R=R^+$ n^2 PHYSICAL HIGGS BOSONS.

$$SU(2n) \rightarrow SU(n) \times SU(n) \times U(1) \xrightarrow{\Phi^0} SU(n)_{\text{DIAG.}}$$

n^2 GAUGE BOSONS BECOME MASSIVE, $M_W^2 = 1/2R^2$

HIGGS MASS $M_h^2 = 1/R^2$

FERMIONSON S^2 EIGENSPINORS:

$$\mathcal{D}_{S^2}^{(p)} \chi_{(j,p)} = \pm M_{j,p} \chi_{(j,p)}$$

$$M_{j,p} = \frac{1}{R} \sqrt{\left(j + \frac{1+p}{2}\right) \left(j + \frac{1-p}{2}\right)}$$

DEGENERACY
 $2j+1$

$$\chi_{(j,p)} = \begin{pmatrix} \chi_{(j,p)}^+ \\ \pm \chi_{(j,p)}^- \end{pmatrix}$$

$$j \geq \frac{1+|p|}{2}$$

 $\chi_{(j,p)}^\pm$ SECTION OF $\mathcal{L}^{p \pm 1}$

$$j = \frac{|p|}{2} - \frac{1}{2} \Rightarrow |p| \text{ ZERO MODES, } |p| \geq 1$$

 $p \geq 1$: p NEGATIVE CHIRALITY MODES $\chi_{(p)}^-$ $p \leq -1$: $|p|$ POSITIVE CHIRALITY MODES $\chi_{(p)}^+$

$$\text{INDEX} (\mathcal{D}^{(p)}) = -p$$

$$\text{ON } X, \quad \{ \Gamma^{\hat{\mu}}, \Gamma^{\hat{\nu}} \} = -2 \hat{g}^{\hat{\mu}\hat{\nu}} \mathbb{1}$$

$$\gamma^{\hat{\mu}} = -\frac{(1+\gamma\bar{\gamma})}{2R} \sigma_+ \quad \gamma^{\bar{\mu}} = \frac{(1+\gamma\bar{\gamma})}{2R} \sigma_-$$

$$\Gamma^{\hat{\mu}} = \gamma^{\hat{\mu}} \otimes \mathbb{1}_2, \quad \Gamma^{\hat{\nu}} = -\gamma^{\hat{\nu}} \otimes \gamma^{\hat{\nu}}, \quad \Gamma^{\bar{\mu}} = \gamma^{\hat{\mu}} \otimes \gamma^{\bar{\mu}}$$

↑
CHIRALITY OPERATOR
ON M , $\gamma^2 = 1$

$\Gamma^{\hat{\mu}}$: $2^{d/2+1} \times 2^{d/2+1}$ MATRICES

Ψ ON X IN FUNDAMENTAL OF $SU(R)$:

$$\Psi^{\pm} = \frac{1}{2} (1 \pm \sigma_3) \Psi^{\pm}, \quad \Psi^{\pm} = \begin{pmatrix} \Psi_0^{\pm} \\ \vdots \\ \Psi_m^{\pm} \end{pmatrix}$$

$$\Psi_i^+ = \sum_{j=1}^{\infty} \sum_{\tau=0}^{2j} \psi_{(j, p_i)_{\tau}}(x) \chi_{(j, p_i)_{\tau}}^+(y)$$

$$\Psi_i^- = \sum_{j=1}^{\infty} \sum_{\tau=0}^{2j} \tilde{\psi}_{(j, p_i)_{\tau}}(x) \chi_{(j, p_i)_{\tau}}^-(y)$$

↑
↓
DIRAC ON Π

ZERO MODE CONTRIBUTIONS:

$$p_i \geq 1, \quad \sum_{\tau=0}^{p_i-1} \tilde{\psi}_{(p_i)_{\tau}}(x) \chi_{(p_i)_{\tau}}^- \quad \text{TO } \Psi^-$$

$$p_i \leq -1 \quad \sum_{\tau=0}^{-p_i-1} \psi_{(p_i)_{\tau}}(x) \chi_{(p_i)_{\tau}}^+$$

DIRAC ACTION ON X :

$$\hat{S}_0 = \int_X d^{d+2}x \sqrt{g} \bar{\Psi}^+ \hat{D}(\mathcal{A}) \Psi$$

\mathcal{A} CONTRIBUTES A TERM:

$$\begin{aligned} & \bar{\Psi}^+ (\bar{\Phi} \gamma \otimes \sigma_- \rightarrow \bar{\Phi}^+ \gamma \otimes \sigma_+) \Psi \\ & = ((\bar{\Psi}^+)^{\dagger}, (\bar{\Psi}^-)^{\dagger}) \begin{pmatrix} \bar{\Phi}^+ \gamma \Psi^- \\ \bar{\Phi} \gamma \bar{\Psi}^+ \end{pmatrix} \end{aligned}$$

INTEGRATE OVER S^2 :

$$S_0 = S_{D_0} + \sum_{i=0}^m S_0^{(p_i)}$$

$$S_{D_0} = \int_M d^d x \sqrt{g} \left\{ \sum_{i=0}^{m_-} \sum_{r=0}^{p_i-1} \tilde{\Psi}_{(p_i)r}^+ \not{D}_M \tilde{\Psi}_{(p_i)r} + \sum_{i=m_+}^m \sum_{r=0}^{p_i-1} \Psi_{(p_i)r}^+ \not{D}_M \Psi_{(p_i)r} \right\}$$

$$\frac{1}{2} \int_M d^d x \sqrt{g} \left\{ \phi_{m_+}^+ \Psi_{(1)}^+ \gamma \hat{\Psi}_{(1)} + \phi_{m_+} \hat{\Psi}_{(1)}^+ \gamma \Psi_{(1)} \right\}$$

$$m_+ = \left\lceil \frac{m+1}{2} \right\rceil, \quad m_- = \left\lfloor \frac{m-1}{2} \right\rfloor$$

ONLY PRESENT
FOR ODD m

- FOR m ODD, ZERO MODES GIVE YUKAWA COUPLINGS TO $\Psi_{(1)}$ AND $\tilde{\Psi}_{(1)}$ ON M .

NON-ZERO MODES:

$$S_D^{(\rho_i)} = \int_M d^d x \sqrt{g} \sum_{j=j_{\min}}^{\infty} \sum_{\tau=0}^{2j}$$

$$\times \left\{ \psi_{(j,\rho_i)\tau}^\dagger \left(\not{D}_n + M_{j,\rho_i} \not{\delta} \right) \psi_{(j,\rho_i)\tau} + \tilde{\psi}_{(j,\rho_i)\tau}^\dagger \left(\not{D}_n + M_{j,\rho_i} \not{\delta} \right) \tilde{\psi}_{(j,\rho_i)\tau} + \frac{g}{2} \left(\psi_{(j,\rho_i)\tau}^\dagger \phi_i^+ \not{\delta} \tilde{\psi}_{(j,\rho_i+2)\tau} + \text{h.c.} \right) \right\}$$

• $\phi_i^0 = v_i$, MASS MATRIX

$$\left(\psi_{(j,\rho_i)\tau}^\dagger, \tilde{\psi}_{(j,\rho_i+2)\tau}^\dagger \right) \begin{pmatrix} M_{j,\rho_i} & g v_i / 2 \\ g v_i / 2 & M_{j,\rho_i+2} \end{pmatrix} \begin{pmatrix} \delta \psi_{(j,\rho_i)\tau} \\ \delta \tilde{\psi}_{(j,\rho_i+2)\tau} \end{pmatrix}$$

$$M_{\pm} = \frac{M_{j,\rho_i} + M_{j,\rho_i+2} \pm \sqrt{(M_{j,\rho_i} - M_{j,\rho_i+2})^2 + g^2 v_i^2}}{2}$$

$$M_{j,\rho_i} \sim 1/R \quad v \sim 1/gR \quad \Rightarrow \quad M_{\pm} \sim 1/R$$

• CHOOSE $\Psi = \begin{pmatrix} \Psi^+ \\ \Psi^- \end{pmatrix}$ TO BE WEYL ON \mathcal{Y}

$$\Rightarrow \Psi^+ \sim \psi^+ \chi^+, \quad \Psi^- \sim \tilde{\psi}^- \chi^-$$

WITH $\psi^+, \tilde{\psi}^-$ WEYL ON M ($\psi^- = \tilde{\psi}^+ = 0$)

IF g_{MN} IS LORENTZIAN \Rightarrow NO DIRECT MASS TERM, ONLY YUKAWA TERMS!

FUZZY SPHERE (MADORE, 1992)

$S^2 \approx CP^1 : \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \in \mathbb{C}^2, \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \approx \lambda \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} : S^2$
 $\forall \lambda \in \mathbb{C} \setminus \{0\}$

CHOOSE λ s.t. $z^1 z^2 = 1 \Rightarrow S^3 \subset \mathbb{C}^2$

$\begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \approx e^{i\delta} \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \Rightarrow S^3 / U(1) \approx SU(2) / U(1) \approx S^2$

FUNCTION $f_L = \sum_{l=0}^L \sum_{m=-l}^l f_l^m Y_m^l(\theta, \phi)$
 $= \underbrace{f_{\alpha_1 \dots \alpha_L}}_{(L+1)^2 \text{ CO-EFFICIENTS}} \bar{z}_{\beta_1} \dots \bar{z}_{\beta_L} z^{\alpha_1} \dots z^{\alpha_L}$

$\left[\frac{\partial}{\partial \bar{z}_\beta}, \bar{z}_\alpha \right] = \delta_\alpha^\beta \quad \xrightarrow{\sigma} \quad [a^\beta, a_\alpha^\dagger] = \delta_\alpha^\beta$

FOCK SPACE : $a^\beta |0\rangle = 0$

MATRIX $(L+1) \times (L+1)$ $\hat{f}_L = f_{\alpha_1 \dots \alpha_L} a_{\beta_1}^\dagger \dots a_{\beta_L}^\dagger |0\rangle \langle 0| a^{\alpha_1} \dots a^{\alpha_L}$

$\bar{z}_\alpha \rightarrow (a_\alpha^\dagger)^L, \quad \frac{\partial}{\partial \bar{z}_\alpha} \rightarrow (a_\alpha)^L$
 $z^\alpha \rightarrow (a^\alpha)^R, \quad \frac{\partial}{\partial z^\alpha} \rightarrow (a_\alpha^\dagger)^R$

STAR PRODUCT

MATRIX MULTIPLICATION $\hat{f}_L \cdot \hat{g}_L \xrightarrow{\sigma} f_L \star g_L$

$[\hat{f}_L, \hat{g}_L] \neq 0 \Rightarrow [f_L, g_L]_\star \neq 0$

FUZZY DIRAC OPERATOR

$$\sigma_{\pm} = \frac{1}{2} (\sigma_1 \pm i \sigma_2), \quad \sigma_- |\Omega\rangle = 0 \quad |\Omega\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

CLIFFORD
VACUUM

DIRAC SPINOR : $\chi = \chi^- |\Omega\rangle + \chi^+ \sigma_+ |\Omega\rangle$

χ^{\pm} SECTIONS OF $\mathcal{L}^{\pm 1/2}$ ($\mathcal{L}^{\pm 1/2}$ FOR SPIN_c)

$$\not{D}_{S^2} = \sigma^+ \bar{\nabla} + \sigma^- \nabla \quad \begin{array}{l} \bar{\nabla}: \chi^- \rightarrow \chi^+ \\ \nabla: \chi^+ \rightarrow \chi^- \end{array}$$

FUZZY SPINORS (GROSSE + PREŠNADLER)

$$\chi^{\pm} \rightarrow \hat{\chi}^{\pm} \left. \begin{array}{l} \rightarrow L \times (L+1) \text{ MATRIX} \\ \rightarrow (L+1) \times L \text{ MATRIX} \end{array} \right\} L = j_{\text{max}} + 1/2$$

$$\begin{array}{l} \bar{\nabla} \rightarrow \bar{K} = (a^{\alpha})^L (a^{\beta})^R \epsilon_{\alpha\beta} \\ \nabla \rightarrow K = (a^{\alpha})^L (a^{\beta})^R \epsilon^{\alpha\beta} \end{array} \quad \begin{array}{l} \bar{K}: \hat{\chi}^- \rightarrow \hat{\chi}^+ \\ K: \hat{\chi}^+ \rightarrow \hat{\chi}^- \end{array}$$

$$\not{D} = \sigma^+ \bar{K} + \sigma^- K$$

INCLUDING MONOPOLES

$$\hat{\chi}^+ : (L - p/2) \times (L + 1 + p/2)$$

$$\hat{\chi}^- : (L + 1 - p/2) \times (L + p/2)$$

$$\not{D} = \sigma^+ \bar{K} + \sigma^- K$$

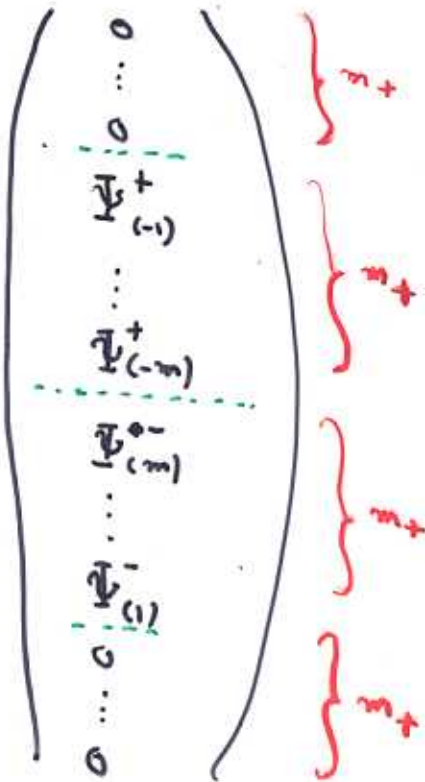
UNIVERSAL!

ZERO MODES : $L_{\text{min}} = \frac{|p|-1}{2}$

$$\begin{array}{l} \hat{\chi}^+_{\alpha_1 \dots \alpha_{|p|-1}} = a^{\alpha_1} \dots a^{\alpha_{|p|-1}} |0\rangle\langle 0|, p \leq -1 \\ \hat{\chi}^-_{\alpha_1 \dots \alpha_{p+1}} = |0\rangle\langle 0| a^{\alpha_1} \dots a^{\alpha_{p+1}}, p \geq 1 \end{array}$$

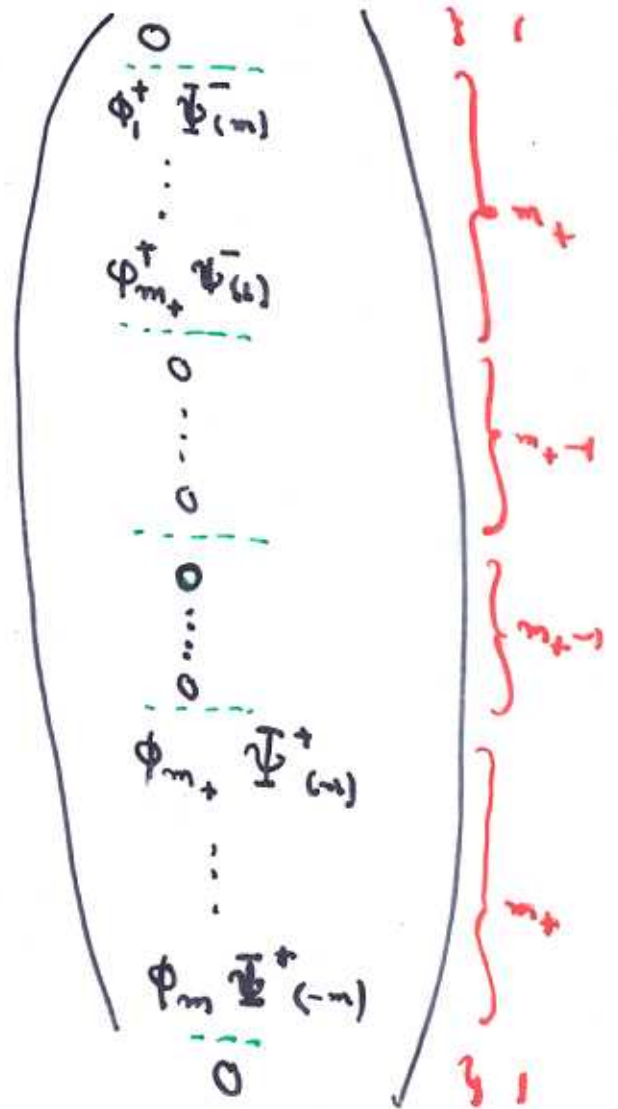
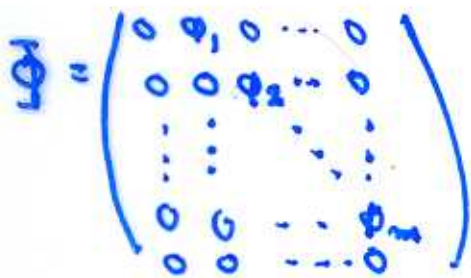
m odd, $m = 2m_+ - 1$, ZERO MODES, Ψ_0

$$\Psi_0 = \begin{pmatrix} \Psi_0^+ \\ \Psi_0^- \end{pmatrix}$$



$$\Psi_0^\dagger = \sum_{i=0}^{\infty} \Psi_i^\dagger (\phi_i)$$

$$(\Psi^\dagger \sigma^+ + \Psi \sigma^-) \Psi_0 =$$



CONCLUSIONS + OUTLOOK

- EQUIVARIANT DIMENSIONAL REDUCTION
 - DYNAMICAL SYMMETRY BREAKING
 - ZERO MODES OF ϕ ACQUIRE YUKAWA COUPLINGS
- OTHER COSETS, K/H (CP^N_R)