Hilbert von Neumann (bi)modules
joint work with
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A Hilbert C^*-module over a C^*-algebra B is a $\mathbb{C}$-vector space E which comes equipped with a *right*- action $E \times B \to E$, and a B-valued inner product

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As may be expected, one needs to know a fair bit of C^*-algebra theory before working with these objects.

von Neumann algebras being C^*-algebras with a distinguished other (σ-*weak*) topology, the existing treatments (e.g., Skeide's) of Hilbert von Neumann modules regard them as Hilbert C^*-modules with additional structure. Then when one gets into dealing with constructions such as tensor-products of bimodules, one finds several stages of abstraction involved - first a 'separation' step involving quotienting out by the radical of the B-valued possibly semi-inner product one gets, then a completion with respect to the norm in E, and finally the von Neumann completion of the result, often ending up with an unrecognisable abstract construct.

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(We prefer to directly rely on the rich structure of von Neumann algebras, the non-commutative analogues of Polish spaces!)

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if $S \subset \mathcal{L}(\mathcal{H}, \mathcal{K})$, then $[S]$ denotes the closure in the SOT (equivalently WOT) of the linear subspace generated by S.

If $S \subset \mathcal{L}(\mathcal{K}, \mathcal{M})$, $T \subset \mathcal{L}(\mathcal{H}, \mathcal{K}), \mathcal{S} \subset \mathcal{H}$, then,

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Definition

A $(1,2)$ *von Neumann corner* is a subset $E \subset \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ satisfying

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Theorem

E is a $(1,2)$ von Neumann corner as above if and only if there exists a von Neumann algebra $M \subset \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ which contains the projections e_i onto the $\mathcal{H}_i$'s such that $E = e_1 M e_2$.

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If A_2 is a von Neumann algebra, a $(1,2)$ von Neumann corner E is called a Hilbert von Neumann A_2-module if there exists a normal isomorphism π_2 of A_2 onto $[E^*E]$. We write $\mathcal{E} = (E, \mathcal{H}_1, (\pi_2, \mathcal{H}_2))$ for the module. The projections $p_1^{(E)} = \bigvee\{q : q \in \mathcal{P}([EE^*])$ and $p_2^{(E)} = \bigvee\{p : p \in \mathcal{P}([E^*E])$ are called the **left-** and **right-support projections of** E.

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A Hilbert von Neumann A_2-module E does indeed admit a right action of A_2 and an A_2-valued inner product thus:

$$x.a_2 = x\pi_2(a_2) \text{ and } \langle x, y\rangle = \pi_2^{-1}(x^*y) \ .$$

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If A_1, A_2 are von Neumann algebras, a a Hilbert von Neumann A_2-module is called a Hilbert von Neumann $A_1 - A_2$-bimodule if there exists a normal homomorphism $\pi_1 : A_1 \to [EE^*]$. We write $\mathcal{E} = (E, (\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$ for the bimodule.

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Lemma (Epd)

If $E \subset \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ is a $(1,2)$ von Neumann corner, and if $x \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ has polar decomposition $x = u|x|$, then

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In order to verify that our definitions agree with those of Skeide, we need to prove that our von Neumann modules satisfy the Riesz lemma, and are hence what he calls *self-dual*; specifically:

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Lemma

(**Riesz Lemma**) *If E is a Hilbert von Neumann A_2-module, and if $f : E \to A_2$ is norm-bounded and satisfies $f(x.a_2) = f(x).a_2 \ \forall a_2 \in A_2$, then $\exists y \in E$ such that $f(x) = y^*x \ \forall x \in E$.*

Riesz lemma (contd.)

Proof.

We are given that f is norm bounded so there exists $K > 0$ such that $\|f(x)\| \leq K\|x\| \ \forall x \in E$. Deduce that if $x \in E$ has polar decomposition $x = u|x|$ and if $\xi \in \mathcal{H}_2$, then

$$\begin{aligned} \|f(x)\xi\| &= \|f(u|x|)\xi\| \\ &= \|f(u)|x|\xi\| \\ &\leq K\||x|\xi\| \\ &= K\|u^*x\xi\| \\ &\leq K\|x\xi\| \ . \end{aligned} \tag{1}$$

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Next choose a sequence $\{\xi_n\} \subset \mathcal{H}_2$ such that $p_2\mathcal{H}_2 = \oplus_n[E^*E\xi_n]$, whence also $p_1\mathcal{H}_1 = \oplus_n[E\xi_n]$. (e.g., $\langle x\xi_m, y\xi_n\rangle = \langle y^*x\xi_m, \xi_n\rangle = 0$ if $n \neq m$.)

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It follows from the previous paragraph and the estimate (1) that $\|f(x)\xi\| \leq K\|x\xi\| \; \forall x \in E, \xi \in \mathcal{H}_2$ and hence that there exists a unique $z \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ such that $z = zp_1$ and

$$z(x\xi) = f(x)\xi \; \forall x \in E, \xi \in \mathcal{H}_2 \; .$$

□

proof (contd.)

Now the definition shows that $zE \subset [E^*E]$ and hence

$$z = zp_2 \in z[EE^*] \subset [zEE^*] \subset [E^*EE^*] = E^*$$

so $y = z^* \in E$ and finally $f(x) = zx = y^*x$. □

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The above version of Riesz' lemma may be used to show that given a Hilbert von Neumann A_2-module $\mathcal{E}$, if $S \subset E$, then

$$S^{\perp\perp} = [SE^*E]$$

and there is no pathology as in the case of Hilbert C^*-modules.

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Given a Hilbert von Neumann $A_1 - A_2$ bimodule $\mathcal{E} = (E, (\pi_1, \mathcal{H}_1), (\pi_2, \mathcal{H}_2))$ and a Hilbert von Neumann $A_2 - A_3$ bimodule $\mathcal{F} = (F, (\rho_2, \mathcal{K}_2), (\rho_3, \mathcal{K}_3))$ there is a Hilbert von Neumann $A_1 - A_3$ bimodule $\mathcal{E} \otimes_{A_2} \mathcal{F} = (E \bigodot F, (\widetilde{\pi_1}, \widetilde{\mathcal{H}_1}), (\rho_3, \mathcal{K}_3))$ which we call their Connes fusion, towards whose definition we head:

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If we directly plunge into the general definition, the elegance of the notion may be missed. To start with, we shall assume that our bimodules are non-degenerate (i.e., $p_i^{(E)} = id_{\mathcal{H}_i}$). We shall give the definition of Connes' fusion in three steps of increasing generality in order to convey the fact that it is actually a 'glorified composition':

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Case 1: Suppose the representations π_2 and ρ_2 are unitarily equivalent, and $u : \mathcal{K}_2 \to \mathcal{H}_2$ is an A_2-linear unitary map. This happens, for instance, if A_2 is a type III factor. Then $E \bigodot F$ consists of the WOT-closed span of the composite opertors

$$x \bigodot y = y \circ u \circ x : \mathcal{K}_3 \to \mathcal{H}_1$$

for $x \in E, y \in F$.

Case 2: Suppose ρ_2 is a multiple of π_2, so that there exists a unitary operator $u : \mathcal{K}_2 \to \mathcal{H}_2 \otimes \mathbb{C}^N$ such that $u\rho_2(a_2) = (\pi_2(a_2) \otimes id_{\mathbb{C}^N})u$. Then $E \bigodot F$ consists of the WOT-closed span of the composite operators

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Case 3: In general, any representation of a von Neumann algebra is unitarily equivalent to a subrepresentation of an infinite (separable) ampliation of any faithful representation. So there exists an isometric A_2-linear operator $u : \mathcal{K}_2 \to \mathcal{H}_2 \otimes \ell^2$, and then $E \bigodot F$ consists of the WOT-closed span of the composite operators

$$x \bigodot y = (y \otimes id_{\ell^2}) \circ u \circ x : \mathcal{K}_3 \to \widehat{\mathcal{H}_1} \subset \mathcal{H}_1 \otimes \ell^2$$

for $x \in E, y \in F$, where $\widehat{\mathcal{H}_1}$ is a suitable subspace of $\mathcal{H}_1 \otimes \ell^2$. To describe this subspace properly, we need a lemma.

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Lemma ($\mathcal{E}_* p$)

If $\mathcal{E}$ is a Hilbert von Neumann $A_1 - A_2$-bimodule, and if $p \in \mathcal{P}(\pi_2(A_2)')$, and if we let q be the projection onto $[Ep\mathcal{H}_2]$, then $q \in \pi_1(A_1)'$ and $xp = qx \ \forall x \in E$; and we shall write $q = \mathcal{E}_* p$.

More on Connes fusion

In the general possibly degenerate case, we observe that as π_2 is a faithful normal representation of A_2 on $p_2^{(E)}\mathcal{H}_2$, there exists a partial isometry $u : \mathcal{K}_2 \to \mathcal{H}_2 \otimes \ell^2$ such that $u^* u = p_1^{(F)}, uu^* \leq p_2^{(E)} \otimes id_{\ell^2}$, and which is A_2-linear, meaning that

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2. Let $E \bigodot F = [\{x \bigodot y : x \in E, y \in F\}], p = uu^*, q = (\mathcal{E} \otimes id_{\ell^2})_* p$ and let $\widetilde{\mathcal{H}_1} = q(\mathcal{H}_1 \otimes \ell^2), \widetilde{\pi}_1 = q(\pi_1(\cdot) \otimes id_{\ell^2})|_{ran\ q}$.

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Definition

Two Hilbert von Neumann $A_1 - A_2$ bimodules, say $\mathcal{E}^{(i)} = (E^{(i)}, (\pi_1^{(i)}, \mathcal{H}_1^{(i)}), (\pi_2^{(i)}, \mathcal{H}_2^{(i)}))$ are said to be isomorphic if there exist A_j-linear unitary operators $u_j : \mathcal{H}_j^{(1)} \to \mathcal{H}_j^{(2)}$ such that $E^{(2)} = u_1 E^{(1)} u_2^*$

In the general possibly degenerate case, we observe that as π_2 is a faithful normal representation of A_2 on $p_2^{(E)}\mathcal{H}_2$, there exists a partial isometry $u : \mathcal{K}_2 \to \mathcal{H}_2 \otimes \ell^2$ such that $u^*u = p_1^{(F)}, uu^* \leq p_2^{(E)} \otimes id_{\ell^2}$, and which is A_2-linear, meaning that

$$u\rho_2(a_2) = (\pi_2(a_2) \otimes id_{\ell^2})u \ ,$$

1. for $x \in E, y \in F$, we define $x \bigodot y$ to be the composite operator
$$\mathcal{K}_3 \stackrel{y}{\to} \mathcal{K}_2 \stackrel{u}{\to} \mathcal{H}_2 \otimes \ell^2 \stackrel{x \otimes id_{\ell^2}}{\longrightarrow} \mathcal{H}_1 \otimes \ell^2$$
2. Let $E \bigodot F = [\{x \bigodot y : x \in E, y \in F\}], p = uu^*, q = (\mathcal{E} \otimes id_{\ell^2})_* p$ and let $\widetilde{\mathcal{H}_1} = q(\mathcal{H}_1 \otimes \ell^2), \widetilde{\pi_1} = q(\pi_1(\cdot) \otimes id_{\ell^2})|_{ran\ q}$.

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It can be shown that up to isomorphism, the Connes fusion $\mathcal{E} \otimes_{A_2} \mathcal{F}$ is independent of the choice of the partial isometry u used in its definition.

1. Any $(1, 2)$ von Neumann corner E can be viewed as a $[EE^*] - [E^*E]$-bimodule; and by replacing $\mathcal{H}_i$ by $p_i\mathcal{H}_i$, we can even assume that the bimodule is **non-degenerate** in the sense that the support projections satisfy $p_i = id_{\mathcal{H}_i}$.

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2. $M_{m\times n}(\mathbb{C})$ is a non-degenerate $M_m(\mathbb{C}) - M_n(\mathbb{C})$-bimodule, just as $\mathcal{L}(\mathcal{K},\mathcal{H})$ is an $\mathcal{L}(\mathcal{H}) - \mathcal{L}(\mathcal{K})$-bimodule.

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3. Any automorphism θ of a von Neumann algebra, M corresponds to a Hilbert von Neumann $M - M$ bimodule $\mathcal{E}_\theta = (Mu_\theta, (id_M, L^2(M)), (id_M, L^2(M))$, where u_θ is the unitary operator on $L^2(M)$ given by $u_\theta\hat{x} = \widehat{\theta(x)}$, where we simply write $L^2(M)$ for $L^2(M,\phi)$ for some faithful normal state ϕ on M, and which satisfies $u_\theta x u_\theta^{-1} = \theta(x)$. It follows fairly easily from the definitions that if ϕ is another automorphism of M, then $\mathcal{E}_\theta \otimes_M \mathcal{E}_\phi = \mathcal{E}_{\theta\circ\phi}$.

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It can also be shown, with a little more work, that $\mathcal{E}_\theta \cong \mathcal{E}_\phi$ if and only if θ and ϕ are inner conjugate (meaning that $\theta(\cdot) = u\phi(\cdot)u^*$ for some $u \in \mathcal{U}(M)$).

4. If $\eta : A_1 \to A_2$ is a unital normal completely positive map, there exists a Hilbert von Neumann $A_1 - A_2$ bimodule $\mathcal{E}_\eta$ whose $(1,2)$-corner E is singly generated - i.e., $E = \pi_1(A_1)V\pi_2(A_2)$ - with the generator V satisfying $V^*\pi_1(a_1)V = \pi_2(\eta(a_2))$. Such a bimodule is unique if some minimal conditions are imposed on it.

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5. If $A_1 \supset A_2$ is a unital inclusion, if ϕ is a faithful normal state on A_1, and if there exists a ϕ-preserving (faithful) normal conditional expectation $\epsilon : A_1 \to A_2$, then the associated $\mathcal{E}_\epsilon$ will satisfy $E^*E = A_2, V^* = E$ and $EE^* = A_0$ where $A_2 \subset A_1 \subset A_0$ is an instance of the Jones construction.

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Actually, for the isomorphism statements asserted, we need to assume that $\mathcal{H}_2 = L^2(A_2, \phi)$ for some faithful normal state ϕ on A_2 and that the bimodule $\mathcal{E}$ is non-degenerrate.

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Finally, to see that our notion of Connes fusion agrees with the classical notion of internal tensor product, one only needs to verify that Connes' fusion satisfies

$$\langle x_1 \bigodot y_1, x_2 \bigodot y_2 \rangle_{\mathcal{K}_3} = \langle y_1, \langle x_1, x_2 \rangle_{\mathcal{H}_2} \cdot y_2 \rangle_{\mathcal{K}_3} ,$$

which is a pleasant little exercise.

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