von Neumann algebras and Free Probability

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1 Non-commutative probability spaces

1.1 Groups and algebras

Groups are described by their repesentations. These in turn are encoded by their group algebras in case the groups are finite, or by their appropriate completions in case the groups are infinite. Thus

$$\Gamma \leftrightarrow \hat{\Gamma} \leftrightarrow \mathbb{C}\Gamma, C^*_{red}(\Gamma), L\Gamma$$

EXAMPLE 1.1.
1. In the case of the permutation group $\Gamma = \Sigma_n$, it is a classical fact that its irreucible representations are parametrised by the set $\mathcal{P}_n$ of partitions of n; thus, $\mathbb{C}\Sigma_n \cong \oplus_{\pi \in \mathcal{P}_n} M_{d_\pi}(\mathbb{C})$.

2. Even for the simplest infinite group $\Gamma = \mathbb{Z}$, there are several natural completions of $\mathbb{C}\mathbb{Z}$ are on offer: thus, we can consider any of

   - $\ell^1(\mathbb{Z})$
   - $C^*_{red}(\mathbb{Z})$ = norm closure, in $B(\ell^2(\mathbb{Z}))$, of span of $\{\lambda_n : n \in \mathbb{Z}\}$, which is isomorphic to the algebra $C(\mathbb{T})$.
   - $L\mathbb{Z}$ = strong closure, in $B(\ell^2(\mathbb{Z}))$, of span of $\{\lambda_n : n \in \mathbb{Z}\}$, which is isomorphic to $L^\infty(\mathbb{T}, \frac{1}{2\pi}d\theta)$.
     Questions regarding the *harmonic analysis* of $\mathbb{Z}$ are treated most naturally in the context of this classical probability space $L^\infty(\mathbb{T}, \frac{1}{2\pi}d\theta)$.

1.2 The von Neumann algebra $L\Gamma$

For a countable group Γ (typically non-commutative), the closures, in the norm- and strong-operator topologies of the algebra of operators generated by the left-regular reresentation λ of Γ in $B(\ell^2(\Gamma))$ are called, respectively, the **reduced group C^*-algebra** $C^*_{red}(\Gamma)$, and the **group von Neumann algebra**

$$L\Gamma = \lambda(\Gamma)'' \ .$$

Incidentally, the commutant, of $L\Gamma$ is the von Neumann algebra generated by the right-regular representation.; thus:

$$(L\Gamma)' = \rho(\Gamma)'' = J\ (L\Gamma)\ J$$

where J is the anti-unitary conjugation on $\ell^2(\Gamma)$ given by $J\xi_t = \xi_{t^{-1}}$.

The counterpart of Lebesgue measure for the *non-commutative probability space* $M = L\Gamma$ is provided by the *positive, faithful, normal, tracial state* - definitions to follow - on M defined by

$$tr(x) = \langle x\xi_1, \xi_1\rangle\ ,\ x \in M$$

1.3 Just definitions

DEFINITION 1.2.

1. *A* **C^*-algebra** *A is a norm closed self-adjoint algebra of operators on a (usually separable) Hilbert space $\mathcal{H}$.*
2. *A* **von Neumann algebra** *M is a strongly closed self-adjoint algebra of operators on a (usually separable) Hilbert space $\mathcal{H}$.*
3. *A linear functional ϕ on a C^*-algebra A is said to be* **positive** *if $\phi(x^*x) \geq 0\ \forall x \in A$.*
4. *A positive linear functional ϕ on a C^*-algebra A is said to be a* **state** *if $\|\phi\| = 1$.*
5. *A state ϕ on a C^*-algebra A is said to be* **faithful** *if $\phi(x^*x) > 0\ \forall\ 0 \neq x \in A$.*
6. *A state ϕ on a von Neumann algebra M is said to be* **normal** *if it is strongly continuous or equivalently if it satisfies the monotone convergence theorem, i.e., it preserves suprema of monotonically increasing uniformly bounded sequences of positive self-adjoint operators.*
7. *A linear functional ϕ on an algebra is said to be a* **trace** *if $\phi(xy) = \phi(yx)\ \forall x, y$.*

1.4 Non-commutative probability spaces

DEFINITION 1.3. *A pair (A, ϕ) of a unital algebra A equipped with a linear functional ϕ satisfying $\phi(1) = 1$ is called a* **non-commutative probability space** - *henceforth NCPS.*

NCPS come in various flavours:

- A may be just an algebra.
- A may be a *-algebra.
- A may be a C^*-algebra, when ϕ is usually a state.
- A may be a von Neumann algebra, when ϕ is usually a normal, preferably tracial (if possible) state.

THEOREM 1.4. *If a von Neumann algebra M admits a unique tracial state, then M has trivial center : i.e., $M \cap M' = \mathbb{C}$. Such an M is called a II_1* **factor** *if it is infinite-dimensional.*

Example: If all non-trivial conjugacy classes of Γ are infiinite, then $L\Gamma$ is a II_1 factor.

1.5 Group factors

- The group $\Sigma_\infty = \cup_{n=1}^{\infty} \Sigma_n$ of permutations of $\mathbb{N}$ which move only finitely many integers is an ICC ('infinite conjugacy class') group, and the associated II_1 factor R is manifestly **hyperfinite** in the sense of being the strong closure of an increasing union of a sequence of finite-dimensional C^*-algebras; this factor is, up to isomorphism, the unique hyperfinite II_1 factor.
- $\Gamma = \mathbb{F}_n,\ n \geq 2$, are clearly ICC groups; and $L\mathbb{F}_n$ is known to not be hyperfinite; the big open problem :

$$L\mathbb{F}_n \cong L\mathbb{F}_m \overset{?}{\Leftrightarrow} n = m \ ?$$

The quest to a possible solution to the above preoblem led Voiculescu to his theory of **free probability**.

2 Free Probability

2.1 Free independence

DEFINITION 2.1. *A family of subalgebras $A_i, i \in I$ of a NCPS (A, ϕ) are said to be* **free** *(or freely independent) if whenever $x_j \in A_{i_j}, 1 \leq j \leq n$ satisfy $i_j \neq i_{j+1}\ \forall 1 \leq j < n$ and $\phi(x_j) = 0 \forall j$, then necessarily also $\phi(x_1 x_2 \cdots x_n) = 0$.*

THEOREM 2.2. *Given a family (A_i, ϕ_i) of NCPS of* the same flavour*, there exixts an NCPS (A, ϕ) also of the same flavour and the following properties:*

- *there exist monomorphisms $\pi_i : A_i \to A$ such that $\phi_i = \phi \circ \pi_i\ \forall i$; and*
- *given homomorphisms $\psi_i : A_i \to B$ for some NCPS (B, τ) (of the same flavour) such that $\tau \circ \psi_i = \phi_i\ \forall i$, there exists a unique morphism $\rho : A \to B$ such that $\rho \circ \pi_i = \psi_i\ \forall i$ and $\tau \circ \rho = \phi$.*

The NCPS (A, ϕ) is unique up to isomorphism and is denoted $(A, \phi) = *_{i \in I}(A_i, \phi_i)$ and is called the **free product** *of the family $\{(A_i, \phi_i) : i \in I\}$.*

EXAMPLE 2.3.

$$L\mathbb{F}_n \cong *^n L\mathbb{Z}$$

2.2 Non-crossing partitions

The study of free probability is intimately connected with the theory of **non-crossing partitions**.

Given a finite subset $S \subset \mathbb{N}$, a partition π of S is said to be *non-crossing* if whenever $i < j$ and $k < l$ belong to distinct classes of π, then neither is $k < i < l < j$ nor is $i < k < j < l$. The collection $NC(S)$ of all such partitions is a lattice with respect to (reverse-) refinement order: $\pi \geq \rho$ if π is coarser than ρ or equivalently, if ρ refines π. The largest element of $NC(S)$ is the trivial partition $1_S = \{S\}$

We will be interested in the **moments** ϕ_n of an NCPS (A, ϕ) given by

$$\phi_n : A^n \to \mathbb{C},\ \phi_n(x_1, \cdots, x_n) = \phi(x_1 x_2 \cdots x_n).$$

Given a set X and a family $\{\phi_n : X^n \to \mathbb{C} | n \in \mathbb{N}\}$ of functions, the **multiplicative extension** of this family is the collection $\{\phi_\pi : X^n \to \mathbb{C} | n \in \mathbb{N}, \pi \in NC([n])\}$ of functions defined by

$$\phi_\pi(x_1, \cdots, x_n) = \prod_{C \in \pi} \phi_{|C|}(x_C : C \in \pi)$$

where we write $[n] = \{1, \cdots, n\}$, and the arguments of $\phi_{|C|}$ are listed in increasing order. (So $\phi_n = \phi_{1_{[n]}}$.)

2.3 Free cumulants

Using the definition and moments to check free indepenence of a family of subalgebras os an NCPS is not easy; but fortunately, the computations sometimes become easier in terms of the so-called **free cumulants** which determine and are determined by the moments.

THEOREM 2.4. 1. *Given a set X and two collections of functions $\{\phi_n : X^n \to \mathbb{C}\}_{n \in \mathbb{N}}$ and $\{\kappa_n : X^n \to \mathbb{C}\}_{n \in \mathbb{N}}$, which are extended multiplicatively, the following conditions are equivalent:*

   (a) $\phi_n = \sum_{\pi \in NC([n])} \kappa_\pi$ *for all* $n \in \mathbb{N}$.

   (b) $\kappa_n = \sum_{\pi \in NC([n])} \mu(\pi, 1_{[n]}) \phi_\pi$ *for all* $n \in \mathbb{N}$.

   (c) $\phi_\tau = \sum_{\pi \in NC([n]), \pi \leq \tau} \kappa_\pi$ *for all* $n \in \mathbb{N}, \pi \in NC([n])$.

   (d) $\kappa_\tau = \sum_{\pi \in NC([n]), \pi \leq \tau} \mu(\pi, \tau) \phi_\pi$ *for all* $n \in \mathbb{N}, \pi \in NC([n])$.

   Here, the symbol μ denotes the Möbius function (to be described shortly) associated to the lattice $NC([n])$.

2. *Given an NCPS (A, ϕ), the functions κ_n associated to the moments - given by $\phi_n(x_1, x_2, \cdots, x_n) = \phi(x_1 x_2 \cdots x_n)$ - are called the free cumulants of the NCPS.*

3. *Subalgebras (A_i, ϕ_i) of an NCPS (A, ϕ) are freely independent if and only if the free cumulants satisfy $\kappa_n(x_1, x_2, \cdots, x_n) = 0$ whenever each x_j comes from some A_{i_j} and at least two i_j's are distinct.*

2.4 Möbius inversion in posets

Given a finite poset (= partially ordered set) X, its **Incidence Algebra** is

$$I(X) = \{f : X \times X \to \mathbb{C} | f(x, y) \neq 0 \Rightarrow x \leq y\}$$

and its *defining function* is

$$\zeta(x,y) = \begin{cases} 1 & if\ x \le y \\ 0 & otherwise \end{cases}$$

First list the members of X so that x is listed before y if $x < y$; then $I(X)$ may be identified with a subalgebra of the algebra of upper triangular matrices, and consequently inhertits a natural algebra structure; clearly $f \in I(X)$ is invertible (i.e., is represented by an invertible matrix) if and only if $f(x,x) \neq 0\ \forall x$. Define the **Möbius function** of X by the element μ of $I(X)$ which is the inverse of ζ. The next fact is a direct consequence of the definitions.

THEOREM 2.5 (Möbius inversion in X). *If $f, g \in I(X)$, the following conditions are equivalent:*

- $f(x,z) = \sum_{x \le y \le z} g(y,z)\ \forall x \in X$ *(or $f = \zeta * g$)*
- $g(x,y) = \sum_{x \le y \le z} \mu(x,y) f(y,z)$ *(or $g = \mu * f$)*

2.5 Examples of Möbius functions

EXAMPLE 2.6. 1. *$X = [n]$ with usual ordering;* $\mu(x,y) = \begin{cases} 1 & if\ x = y \\ -1 & if\ x = y-1 \\ 0 & otherwise \end{cases}$

2. *$X = [n], d \le k \Leftrightarrow d|k$;*
$$\mu(x,y) = \begin{cases} 1 & if\ x = y \\ (-1)^k & if\ y/x\ \textit{is (square-free, and) the product of}\ k\ \textit{distinct primes} \\ 0 & otherwise \end{cases}$$

3. *$X = 2^S$ is the set of all subsets of a set S, ordered by inclusion;*
$$\mu(E,F) = \begin{cases} 1 & if\ E = F \\ (-1)^{|F \setminus E|} & if\ E \subset F \\ 0 & otherwise \end{cases}$$

References

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[2] D. V. Voiculescu, K. J. Dykema, A. Nica, Free random variables, CRM Monograph Series, 1992.