On a tensor-analogue of the Schur product

K. Sumesh and V.S. Sunder
Institute of Mathematical Sciences
Chennai 600113
INDIA
email: sumeshkpl@gmail.com, sunder@imsc.res.in

October 1, 2015

Abstract

We consider the *tensorial Schur product* $R \circ^{\otimes} S = [r_{ij} \otimes s_{ij}]$ for $R \in M_n(\mathcal{A}), S \in M_n(\mathcal{B})$, with $\mathcal{A}, \mathcal{B}$ unital C^*-algebras, verify that such a 'tensorial Schur product' of positive operators is again positive, and then use this fact to prove (an apparently marginally more general version of) the classical result of Choi that a linear map $\phi : M_n \to M_d$ is completely positive if and only if $[\phi(E_{ij})] \in M_n(M_d)^+$, where of course $\{E_{ij} : 1 \leq i, j \leq n\}$ denotes the usual system of matrix units in $M_n (:= M_n(\mathbb{C}))$. We also discuss some other corollaries of the main result.

1 The result

We start with some notation: (We assume, for convenience, that all our C^*-algebras are unital.) We denote an element of a matrix algebra by capital letters, such as R, and denote its entries by either $[R]_{ij}$ or the corresponding lower case letter r_{ij}. This is primarily because $[R^*]_{ij} = (r_{ji})^* \neq [R^*]_{ji}$!

Definition 1.1. *1. If $\mathcal{A}, \mathcal{B}$ are C^*-algebras, and $\phi : M_n \to \mathcal{B}$ is a positive map, define $\phi_{\mathcal{A}} : A \otimes M_n \to \mathcal{A} \otimes_{alg} \mathcal{B}$ (where $\mathcal{A} \otimes_{alg} \mathcal{B}$ denotes the algebraic tensor product of $\mathcal{A}$ and $\mathcal{B}$) by $\phi_{\mathcal{A}} = id_{\mathcal{A}} \otimes \phi$.*

2. *If* $A = [a_{ij}] \in M_n(\mathcal{A}), B = [b_{ij}] \in M_n(\mathcal{B})$*, define* $A \circ^{\otimes} B = [a_{ij} \otimes b_{ij}] \in M_n(\mathcal{A} \otimes_{alg} \mathcal{B})$*.*

For later use, we isolate a lemma, whose elementary verification we omit.

LEMMA 1.2. *The map* $\pi : M_n(\mathcal{A}) \otimes M_k(\mathbb{C}) \to M_{nk}(\mathcal{A})$ *defined by*

$$[\pi(R \otimes C)]_{i\alpha, j\beta} = c_{\alpha\beta} r_{ij} \tag{1.1}$$

is a C^**-algebra isomorphism for any* C^**-algebra* $\mathcal{A}$*; in the sequel, we shall simply use this* π *to make the identification* $M_n(\mathcal{A}) \otimes M_k(\mathbb{C}) = M_{nk}(\mathcal{A})$*.*

PROPOSITION 1.3.

$$R \in M_n(\mathcal{A})^+, S \in M_n(\mathcal{B})^+ \Rightarrow R \circ^{\otimes} S \in M_n(\mathcal{C})^+, \tag{1.2}$$

where $\mathcal{C}$ *denotes - here and in the rest of this short note - any* C^**-algebra containing* $\mathcal{A} \otimes_{alg} \mathcal{B}$*. In particular,*

$$\sum_{i,j=1}^{n} r_{ij} \otimes s_{ij} \in M_n(\mathcal{C})^+. \tag{1.3}$$

Proof. To deduce eqn. (1.3) from eqn. (1.2), we let $1_n \in M_{n\times 1}(\mathcal{C})^+$ be the $n \times 1$ column-vector with all entries equal to $1_{\mathcal{A}} \otimes 1_{\mathcal{B}}$, and note that

$$\sum_{i,j=1}^{n} r_{ij} \otimes s_{ij} = 1_n^*(R \circ^{\otimes} S)1_n.$$

Now for the slightly less immediate eqn. (1.2). By assumption, $R \otimes S \in M_{n^2}(\mathcal{C})^+$.

In the sequel all the variables i, j, k, l, p, q will range over the set $\{1, 2, ..., n\}$ and we shall simply write $\sum_k$ for $\sum_{k=1}^n$.

Now define $V \in M_{n\times n^2}(\mathcal{C})$ by

$$[V]_{i,pq} = \delta_{pi}\delta_{qi}(1_A \otimes 1_B).$$

Then,

$$\begin{aligned} [V(R \otimes S)V^*]_{ij} &= \sum_{p,q,k,l} [V]_{i,pq}[R \otimes S]_{pq,kl}[V^*]_{kl,j} \\ &= [R \otimes S]_{ii,jj} \\ &= r_{ij} \otimes s_{ij} \end{aligned}$$

and so,

$$V(R \otimes S)V^* = R \circ^{\otimes} S.$$

The proof of the Proposition is complete. $\square$

REMARK 1.4. *Note that the proof shows that* $R \circ^{\otimes} S \in M_n(\mathcal{A} \otimes_{alg} \mathcal{B})$.

The classical result of Choi alluded to in the abstract is the equivalence 2.$\Leftrightarrow$3. in the following Corollary, for the case $\mathcal{B} = M_d$ (see [1]).

COROLLARY 1.5. *The following conditions on a linear map* $\phi : M_n \to \mathcal{B}$ *are equivalent:*

1. *For any* C^**-algebra* $\mathcal{A}$*, the map* $\phi_{\mathcal{A}}(:= id_{\mathcal{A}} \otimes \phi) : \mathcal{A} \otimes M_n \to \mathcal{C}$ *is a positive map for any* C^**-algebra* $\mathcal{C}$ *as in Proposition 1.3.*

2. *The map* ϕ *is CP.*[1]

3. $[\phi(E_{ij})] \in M_n(\mathcal{B})^+$.

Proof. We only prove the non-trivial implication 3. $\Rightarrow$ 1. if $R \in (\mathcal{A} \otimes M_n)^+ = M_n(\mathcal{A})^+$, and if $R = [r_{ij}]$, then

$$\begin{aligned}
\phi_{\mathcal{A}}(R) &= (id_{\mathcal{A}} \otimes \phi)(\sum_{ij} r_{ij} \otimes E_{ij}) \\
&= \sum_{ij} r_{ij} \otimes \phi(E_{ij}) \\
&\in M_n(\mathcal{C})^+,
\end{aligned}$$

by eqn. (1.3). $\square$

COROLLARY 1.6. *Let* $R \in M_n(\mathcal{A})^+$*. Then the map* $M_n(\mathcal{B}) \ni S \stackrel{L_R}{\to} R \circ^{\otimes} S \in M_n(\mathcal{C})$ *is CP. In particular* $R \in M_n^+ \Rightarrow M_n \ni S \to R \circ S \in M_n$ *is also CP.*

Proof. To avoid confusion, we use Greek letters α, β etc., to denote elements of $\{1, 2, \cdots, k\}$ and English letters i, j etc. to denote elements of $\{1, 2, \cdots, n\}$. Suppose $[\hat{S}] \in M_{kn}(\mathcal{B})^+ = M_k(M_n(\mathcal{B}))^+$ is given by $[\hat{S}]_{\alpha i, \beta j} := [S_{\alpha,\beta}]_{i,j}$ (see Lemma 1.2), where of course $S_{\alpha\beta} \in M_n(\mathcal{B}) \forall \alpha, \beta)$. Let $J_k \in M_k$

[1] For an explanation of terms like CP (= completely positive) and operator system, the reader may consult [3], for instance.

be (the all 1 matrix) given by $[J_k]_{\alpha\beta} = 1 \ \forall \alpha, \beta$. Then we see that $J_k \geq 0$ (in fact J_k/k is a projection) and so,

$$\begin{aligned}
[L_R(S_{\alpha\beta})] &= [[r_{ij} \otimes [S_{\alpha\beta}]_{ij}]] \\
&= [[[J_k]_{\alpha\beta} r_{ij} \otimes [S_{\alpha\beta}]_{ij}]] \\
&= [(J_k \otimes R)]_{\alpha i,\beta j} \otimes [\hat{S}]_{\alpha i,\beta j} \quad \text{(see Lemma 1.2)} \\
&= (J_k \otimes R) \circ^{\otimes} \hat{S} \\
&\geq 0,
\end{aligned}$$

by Proposition 1.3 applied with R, n, S there replaced by $J_k \otimes R, kn, \hat{S}$, since $[J_k \otimes R]_{\alpha i,\beta j} = r_{ij} \ \forall \alpha, \beta$. The second statement of the Corollary is just the specialisation of the first statement to $\mathcal{A} = \mathcal{B} = \mathbb{C}$. $\square$

Remark 1.7. *The special case* $n = 1$ *of Corollary 1.6 perhaps merits singling out: If* $r \in \mathcal{A}^+$*, then the map* $\mathcal{B} \ni s \stackrel{L_r}{\to} r \otimes s \in \mathcal{C}$ *is CP.*

Remark 1.8. *It should be clear that there is a 'right' version of all the 'left' statements discussed above.*

The proofs suggest that these results might well admit formulations in the language of operator systems; however, we suspect that such 'generalisations' will follow from nuclearity of M_n and the flexibility in the choice of $\mathcal{C}$ in our formulation, in view of the Choi-Effros theorem (see [2]).

References

[1] M. D. Choi, *Completely positive linear maps on complex matrices*, Linear Algebra and Appl. **10** (1975), 285–290.

[2] M. D. Choi and E. G. Effros, *Injectivity and operator spaces*, J. of Functional Analysis **24** (1977), no. 2, 156–209.

[3] G. Pisier, *Introduction to Operator Space Theory.* LMS Lecture Note Series 294, Cambridge University Press, 2003.