Spectra of principal graphs

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Abstract

We show that if the adjacency matrices of the two principal graphs of a finite index subfactor are regarded as (necessarily bounded, self-adjoint) operators on the ℓ^2 spaces over their vertex sets, then their spectral measures, when restricted to the complement of $\{0\}$, are mutually absolutely continuous. In particular, for a finite-depth subfactor, the two matrices have the same sets of non-zero eigenvalues.

1 Introduction

We gather together the notation and terminology that we will need, in order to prove the result stated in the abstract. In the sequel, we shall follow standard notation from subfactor theory - as in [JS], say.

Suppose $R_0 \subset R_1$ is a subfactor of finite Jones index; i.e., this is an inclusion of II_1 factors such that $dim_{R_0} L^2(R_1) < \infty$. For $i = 0, 1$, let us write $\alpha_{i,1-i}$ to denote the bimodule $_{R_i} L^2(R_1)_{R_{1-i}}$ which has finite left- and right- dimensions under the assumption of finite index. (In fact, this can be taken as the definition of finiteness of index.) Then the so-called contragredient bimodule of α_{01} is nothing but α_{10}; i.e., $\overline{\alpha_{01}} = \alpha_{10}$.

For $i, j \in \{0, 1\}$, let us write B_{ij} for the set of isomorphism classes of irreducible $R_i - R_j$ bimodules contained in tensor products of the form $\alpha_{i,1-i} \otimes_{R_{1-i}} \alpha_{1-i,i} \otimes_{R_i} \cdots \otimes_{R_{1-j}} \alpha_{1-j,j}$, where the above string of bimodules must start with $\alpha_{i,1-i}$ and end with $\alpha_{1-j,j}$, but is allowed to be arbitrarily long provided its length is of the same parity as $(j - i)$.

The so-called pair of principal graphs of the subfactor is (Γ_0, Γ_1), where, for $p = 0, 1$, the graph Γ_p is the bipartite graph with vertex set $B_{0p} \coprod B_{1p}$, with the number of edges joining γ_{0p} and γ_{1p} being given by the multiplicity $\langle \alpha_{10} \otimes_{R_0} \gamma_{0p}, \gamma_{1p} \rangle$ with which the bimodule γ_{1p} features in the tensor product $\alpha_{10} \otimes_{R_0} \gamma_{0p}$. (Above, and in the sequel, we shall use the symbol $\coprod$ to indicate 'disjoint union'. And we shall adopt the 'subscript convention' whereby a symbol such as γ_{ij} will always denote an element of B_{ij}; the only possible exception to this rule stems from the possibility that in case our subfactor is not irreducible, then $\alpha_{i,1-i}$ is not one element in $B_{i,1-i}$, but a direct sum of finitely many members of $B_{i,1-i}$. Further, all subscripts, such as i, j, p will run over the set $\{0, 1\}$.)

To any graph Γ, with vertex set $V = V(\Gamma)$, is associated its *adjacency matrix* $A = A(\Gamma)$, which has rows and columns indexed by $V(\Gamma)$, and is the symmetric matrix (with non-negative integral entries), given by $A(u, v)$ equal to the number of edges joining u and v.

The subfactor $R_0 \subset R_1$ is said to have finite depth precisely

when each B_{ij} is a finite set, i.e, if the graphs Γ_p are both finite. (In fact, the finiteness of either one of them implies that of the other.) Even in the infinite-depth case, it is true - see Remark 2.2(c) - that the matrix $A_p = A(\Gamma_p)$ defines a bounded (self-adjoint) operator on $\mathcal{L}(\ell^2(V(\Gamma_p)))$.

Henceforth the symbol Δ will denote a Borel set in $\mathbb{R}$ such that $0 \notin \Delta$.

DEFINITION 1.1 *If A and B are self-adjoint operators on Hilbert space, we shall write $A \sim_0 B$ if the following condition is satisfied:*

$$1_\Delta(A) \neq 0 \Leftrightarrow 1_\Delta(B) \neq 0 \quad \forall \Delta .$$

Above, and elsewhere, the symbol $1_\Delta(A)$ denotes the spectral projection for the self-adjoint operator corresponding to the set Δ; the notation is natural, in the sense of the measurable functional calculus for self-adjoint operators. It should be obvious that $\sim_0$ is an equivalence relation. Notice that if the definition above is relaxed to only considering open sets not containing 0, then the corresponding equivalence relation would just be: $sp(A)\backslash\{0\} = sp(B)\backslash\{0\}$, where the symbol sp denotes 'spectrum'.

This paper is devoted to the proof of the following theorem:

THEOREM 1.2 *With the foregoing notation, we have:*

$$A_0 \sim_0 A_1 .$$

Some immediate consequences of this theorem are listed in the following corollary, whose proof is disscussed after the proof of the theorem.

COROLLARY 1.3 *With A_0 and A_1 as above, we have:*
(i) with the only possible exception of 0, the operators A_p have the same spectrum;
(ii) when restricted to $\mathbb{R}\backslash\{0\}$, the spectral measures of the A_p's are mutually absolutely continuous;
(iii) with the only possible exception of 0 and isolated nonzero eigenvalues of infinite multplicity, the operators A_p have the same essential spectrum.

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Finally, in the case of Haagerup's $\frac{5+\sqrt{13}}{2}$ subfactor - see [H] and [AH] - both the numbers $\pm\sqrt{2}$ are eigenvalues of multiplicity 1 for one of the operators and multiplicity 2 for the other, thereby showing that the two self-adjoint operators can have different spectral multiplicity functions.

2   Proof of the main result

The possibility of forming tensor-products and contragredients of bimodules endows the structure of an an involutive algebra on the $\mathbb{C}$-vector space $\mathcal{A}$ with basis given by $\coprod_{i,j=0,1} B_{ij}$. We shall need this M_2-*graded fusion algebra* in what follows, so we spell out the desired facts as a proposition for convenience of future reference. We omit the proof of this well-known fact; the interested reader might wish to consult [EK], for instance.

PROPOSITION 2.1 *Let $\mathcal{A}$ be the $\mathbb{C}$-vector space with basis $\coprod_{i,j=0,1} B_{ij}$. Define*

$$\gamma_{ij}\gamma_{kl} \;=\; \left\{ \begin{array}{cc} 0 & \text{if } j \neq k \\ \sum_{\gamma_{il}} \langle \gamma_{ij} \otimes_{R_j} \gamma_{jl}, \gamma_{il} \rangle \gamma_{il} & \text{if } j = k \ ; \end{array} \right.$$

and define $\overline{\gamma_{ij}}$ to be that unique $\gamma_{ji} \in B_{ji}$ which is isomorphic to the contragredient of the bimodule γ_{ij}. Then,

(i) $\mathcal{A}$ is an associative involutive algebra;

(ii) $\langle \gamma_{ij} \otimes_{R_j} \gamma_{jl}, \gamma_{il} \rangle \;=\; \langle \overline{\gamma_{ij}} \otimes_{R_i} \gamma_{il}, \gamma_{jl} \rangle$; and

(iii) there exist elements $1_i \in B_{ii}$, $i = 0, 1$, such that

$$1_i \gamma_{ij} = \gamma_{ij} \;=\; \gamma_{ij} 1_j \ \forall \gamma_{ij} \ ;$$

and in particular, the element $1 = 1_0 + 1_1$ is a multiplicative identity for $\mathcal{A}$.

When convenient, we shall (i) write $\mathcal{B} = \coprod_{i,j=0,1} B_{ij}$; (ii) denote the typical elements of $\mathcal{B}$ by the symbols X, Y, Z; and (iii) for $X, Y, Z \in \mathcal{B}$, write N^Z_{XY} for the 'fusion coefficients' defined by the equation $XY = \sum_{Z \in \mathcal{B}} N^Z_{XY} Z$ (in $\mathcal{A}$).

We first list some easy consequences of Proposition 2.1.

REMARK 2.2 (a)

$$\sum_{p=0}^{1} N_{XY}^{1_p} \;=\; \sum_{p=0}^{1} N_{\bar{X}1_p}^{Y} \;=\; \delta_{\bar{X},Y} \;,\; \forall X, Y \in \mathcal{B} \; . \qquad (2.1)$$

(b) We shall write $\mathcal{A} \ni x \mapsto L_x \in End_{\mathbb{C}}(\mathcal{A})$ to denote the left regular representation (defined by $L_x(y) = xy$). The map L is one-to-one since $\mathcal{A}$ has an identity.

Equip $\mathcal{A}$ with an inner product so that $\mathcal{B}$ is an orthonormal set. Then the Hilbert space completion of $\mathcal{A}$ with respect to this inner product is naturally identified with the space $\ell^2(\mathcal{B})$ of square-summable functions on the set $\mathcal{B}$. We shall write $\mathcal{H}_{ij}$ for the Hilbert subspace which has B_{ij} as an orthonormal basis.

Notice that $End_{\mathbb{C}}(\mathcal{A})$ is naturally identified with 'column-finite' matrices (with rows and columns indexed by $\mathcal{B}$) - whereby $T \in End_{\mathbb{C}}(\mathcal{A})$ corresponds to the matrix $((t_{YZ}))_{Y,Z \in \mathcal{B}}$ given by $t_{YZ} = \langle T(Z), Y \rangle \; \forall Y, Z \in \mathcal{B}$.

(c) Observe that if $X \in \mathcal{B}$, then

$$\begin{aligned}
(L_{\bar{X}})_{YZ} &= \langle \bar{X}Z, Y \rangle \\
&= N_{\bar{X}Z}^{Y} \\
&= N_{XY}^{Z} \qquad \text{(by Proposition 2.1(ii)} \\
&= \langle XY, Z \rangle \\
&= (L_X)_{ZY} \; .
\end{aligned}$$

Thus if $X \in \mathcal{B}$, then the matrix of $L_{\bar{X}}$ is the transpose of the matrix of L_X; and so, if $x \in \mathcal{A}$, it follows that the matrix of $L_{\bar{x}}$ is the adjoint (= conjugate-transpose) of the matrix of L_x. In particular, each $L_x, x \in \mathcal{A}$ is represented by a matrix which is both row-finite and column-finite.

We now outline a proof of the fact that L_x extends uniquely to a bounded operator, for each $x \in \mathcal{A}$. It suffices to check this when $x \in \mathcal{B}$. So, suppose $x = \gamma \in \mathcal{B}_{pq}$. A little thought reveals that it suffices to verify that the matrix - call it T - of $L_{\gamma\bar{\gamma}}$ represents a bounded operator. Notice that $\gamma\bar{\gamma} \in \mathcal{H}_{pp}$ and that the matrix of T satisfies: $t_{YZ} = 0$ unless both Y and Z belong to B_{pj} for $j = 0, 1$. So, it suffices to check that if T_j is

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the (principal) submatrix of T obtained by restricting the rows and columns to B_{pj}, then T_j is bounded for $j = 0, 1$. Now, it is a fact (see [JS], for instance) that if P, Q are II_1 factors, if $\mathcal{H}_1$ is a $P - Q$ bimodule and $\mathcal{H}_2$ is a left Q-module, then $dim_P(\mathcal{H}_1 \otimes_Q \mathcal{H}_2) = dim_P(\mathcal{H}_1)\, dim_Q(\mathcal{H}_2)$. It follows quite easily from this that if we define the vector v_j by $v_j(Y) = dim_{R_p}(Y)$ for $Y \in B_{pj}$, then v_j is a vector with strictly positive entries (which is not in $\ell^2(B_{pj})$ if B_{pj} is infinite) such that $Tv_j = \lambda v_j$, where $\lambda = dim_{R_p}(\gamma \otimes_{R_q} \bar{\gamma})$. However, a symmetric matrix with a strictly positive eigenvector is known - see [P] - to define a bounded operator on the ℓ^2 space, with norm at most equal to the eigenvalue λ. (A formulation of a more general statement, called the 'Schur test', may be found in [HS].)

Thus, we see that $x \mapsto L_x$ defines a *-homomorphism from $\mathcal{A}$ into $\mathcal{L}(\ell^2(\mathcal{B}))$.

(d) Let $M = (\{L_x : x \in \mathcal{A}\})''$ denote the von Neumann algebra completion of $\mathcal{A}$ (in $\mathcal{L}(\ell^2(\mathcal{B}))$). It is clear that 1 is a cyclic vector for M, since $M1 \supset \mathcal{B}$. On the other hand, we may argue exactly as in (c) that for each $X \in \mathcal{B}$, there exists a unique operator $R_X \in \mathcal{L}(\ell^2(\mathcal{B}))$ such that $R_X(Y) = YX$ for all $X \in \mathcal{B}$. It should be obvious that $R_X \in (\{L_x : x \in \mathcal{A}\})' = M'$; and that hence the vector 1 is also a cyclic vector for M'; consequently, (see [JS], for instance) we see that 1 is a cyclic and separating vector for M. Hence the functional defined by

$$M \ni x \; \mapsto \; \tau(x) = \langle x(1), 1 \rangle$$

is seen to be a faithful positive normal linear functional. Actually, it turns out that $\frac{1}{2}\tau$ is a faithful normal tracial state on M, but we will not need this, and shall say no more about it. $\square$

In what follows, if V_0, V_1 are sets and if a $V_0 \times V_1$ matrix, say G, represents a bounded operator from $\ell^2(V_1)$ to $\ell^2(V_0)$, then, we will use the same symbol to represent this operator as well. So, the symbol G^* will represent the conjugate-transpose matrix as well the adjoint operator of G.

LEMMA 2.3 *Suppose Γ is a bipartite graph, with the associated partition of its vertex set being given by $V = V_0 \coprod V_1$. Then,*

with respect to this decomposition of the vertex set, the adjacency matrix $A = A(\Gamma)$ clearly has the decomposition

$$A = \begin{bmatrix} 0 & G \\ G^* & 0 \end{bmatrix},$$

where G is the $V_0 \times V_1$ matrix defined by setting $G(v_0, v_1)$ equal to the number of edges joining v_0 and v_1. Assume that A induces a bounded (necessarily self-adjoint) operator on $\ell^2(V)$.

For any $0 \notin \Delta \in \mathcal{B}_\mathbb{R}$, we have:

$$1_\Delta(A) \neq 0 \;\Leftrightarrow\; 1_{\Delta^2}(GG^*) \neq 0 \;\Leftrightarrow\; 1_{\Delta^2}(G^*G) \neq 0, \qquad (2.2)$$

where we write Δ^2 for $\{\lambda^2 : \lambda \in \Delta\}$.

Proof: To start with, if $T = W(T^*T)^{\frac{1}{2}}$ is the (left) polar decomposition of a bounded operator T between Hilbert spaces, it is then true that (a) the initial and final spaces of the partial isometry W - call them $\mathcal{M}$ and $\mathcal{N}$ - satisfy $\mathcal{M} = ran(1_{\mathbb{R}\setminus\{0\}}(T^*T))$ and $\mathcal{N} = ran(1_{\mathbb{R}\setminus\{0\}}(TT^*))$, respectively; and (b) when regarded as a unitary operator of $\mathcal{M}$ onto $\mathcal{N}$, the operator $W|_\mathcal{M}$ establishes a unitary equivalence between the operators $(T^*T)|_\mathcal{M}$ and $(TT^*)|_\mathcal{N}$. In particular, it follows that for every $\Delta \in \mathcal{B}_\mathbb{R}$,

$$1_\Delta(G^*G) \neq 0 \;\Leftrightarrow\; 1_\Delta(GG^*) \neq 0. \qquad (2.3)$$

The next fact we need is that if T is a self-adjoint operator on a Hilbert space, if $\Delta \subset (0, \infty)$, and if we write $-\Delta = \{-\lambda : \lambda \in \Delta\}$, then

$$1_{-\Delta \coprod \Delta}(T) = 1_{\Delta^2}(T^2). \qquad (2.4)$$

(This is a simple consequence of the spectral theorem and a 'change of variable' formula in measure theory, whose proof we omit.)

Observe next that if

$$U = \begin{bmatrix} id_{\ell^2(V_0)} & 0 \\ 0 & -id_{\ell^2(V_1)} \end{bmatrix},$$

then U is unitary and $UAU^* = -A$. Hence A and $-A$ are unitarily equivalent, and in particular,

$$1_\Delta(A) \neq 0 \;\Leftrightarrow\; 1_{-\Delta}(A) \neq 0 \;\Leftrightarrow\; 1_{-\Delta \cup \Delta}(A) \neq 0. \qquad (2.5)$$

Let us write $\mathbb{R}^+ = [0, \infty), \mathbb{R}^- = (-\infty, 0]$; and let $\Delta_\pm = \Delta \cap \mathbb{R}^\pm$; then $\Delta = \Delta_+ \coprod \Delta_-$ and $\Delta^2 = \Delta_+^2 \cup \Delta_-^2$; we see therefore that

$$1_{\Delta^2}(T) \leq 1_{\Delta_+^2}(T) + 1_{\Delta_-^2}(T) \leq 2 \, 1_{\Delta^2}(T) \qquad (2.6)$$

for any self-adjoint operator T.

Finally conclude that

$$
\begin{aligned}
1_\Delta(A) \neq 0 \;\; & \Leftrightarrow \;\; 1_{\Delta_+}(A) + 1_{\Delta_-}(A) \neq 0 \\
& \Leftrightarrow \;\; 1_{\Delta_+}(A) \neq 0 \text{ or } 1_{\Delta_-}(A) \neq 0 \\
& \Leftrightarrow \;\; 1_{-\Delta_+ \coprod \Delta_+}(A) \neq 0 \text{ or } 1_{-\Delta_- \coprod \Delta_-}(A) \neq 0 \;\; \text{(by (2.5))} \\
& \Leftrightarrow \;\; 1_{\Delta_+^2}(A^2) \neq 0 \text{ or } 1_{\Delta_-^2}(A^2) \neq 0 \;\; \text{(by (2.4))} \\
& \Leftrightarrow \;\; 1_{\Delta_+^2}(A^2) + 1_{\Delta_-^2}(A^2) \neq 0 \\
& \Leftrightarrow \;\; 1_{\Delta^2}(A^2) \neq 0 \;\; \text{(by (2.6))} \\
& \Leftrightarrow \;\; 1_{\Delta^2}\left(\begin{bmatrix} GG^* & 0 \\ 0 & G^*G \end{bmatrix} \right) \neq 0 \\
& \Leftrightarrow \;\; \begin{bmatrix} 1_{\Delta^2}(GG^*) & 0 \\ 0 & 1_{\Delta^2}(G^*G) \end{bmatrix} \neq 0 \; ;
\end{aligned}
$$

and the desired conclusion follows now from equation (2.3) and the above implications. $\qquad\square$

Proof of Theorem 1.2: For convenience, let us simply write α for what we denoted earlier by α_{01}; then $\overline{\alpha}$ is what we earlier denoted by α_{10}; further, for $p = 0, 1$, if we write A_p for the $(B_{0p} \coprod B_{1p}) \times (B_{0p} \coprod B_{1p})$ matrix and G_p for the $B_{0p} \times B_{1p}$ matrix which are associated to the bipartite graph Γ_p as in Lemma 2.3, we see that

$$
\begin{aligned}
G_p(\gamma_{0p}, \gamma_{1p}) \;\; & = \;\; \langle \overline{\alpha} \otimes_{R_0} \gamma_{0p}, \gamma_{1p} \rangle \\
& = \;\; \langle \alpha \otimes_{R_0} \gamma_{1p}, \gamma_{0p} \rangle \\
& = \;\; \langle L_\alpha(\gamma_{1p}), \gamma_{0p} \rangle \; ;
\end{aligned}
$$

and so G_p is just the restricted operator $L_\alpha|_{\mathcal{H}_{1p}} : \mathcal{H}_{1p} \to \mathcal{H}_{0p}$; and so $G_p^* = L_{\overline{\alpha}}|_{\mathcal{H}_{0p}} : \mathcal{H}_{0p} \to \mathcal{H}_{1p}$. Hence, we deduce from Lemma 2.3 that

$$1_\Delta(A_p) \neq 0 \;\Leftrightarrow\; 1_{\Delta^2}(L_{\overline{\alpha}\alpha}|_{\mathcal{H}_{1p}}) \neq 0 \;\Leftrightarrow\; 1_{\Delta^2}(L_{\alpha\overline{\alpha}}|_{\mathcal{H}_{0p}}) \neq 0 \; . \quad (2.7)$$

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Notice next that $L_{\overline{\alpha}\alpha} = L_{\overline{\alpha}}L_\alpha$ leaves each of the subspaces $\mathcal{H}_{10}, \mathcal{H}_{11}$ invariant, and is identically 0 on $\mathcal{H}_{00} \oplus \mathcal{H}_{01}$. Thus if we let P_{ij} denote the orthogonal projection onto $\mathcal{H}_{ij}$, then $P_{00} + P_{01} \leq 1_{\{0\}}(L_{\overline{\alpha}\alpha})$.

We shall now prove that

$$L_{\overline{\alpha}\alpha}|_{\mathcal{H}_{11}\oplus\mathcal{H}_{10}} \sim_0 L_{\overline{\alpha}\alpha}|_{\mathcal{H}_{11}} . \tag{2.8}$$

For typographical convenience, let us write $T = L_{\overline{\alpha}\alpha}|_{\mathcal{H}_{11}\oplus\mathcal{H}_{10}}$, and $T_1 = T|_{\mathcal{H}_{11}}$. Thus, we need to show that, for all Δ,

$$1_\Delta(T) \neq 0 \Leftrightarrow 1_\Delta(T_1) \neq 0 .$$

Fix $\Delta \in \mathcal{B}_\mathbb{R}$ such that $0 \notin \Delta$. Since $1_\Delta(T_1) = (P_{11}1_\Delta(T))|_{\mathcal{H}_{11}} = (1_\Delta(T)P_{11})|_{\mathcal{H}_{11}}$, we always have:

$$1_\Delta(T_1) \neq 0 \Rightarrow 1_\Delta(T) \neq 0 .$$

So we need to verify that if $Q = 1_\Delta(T) \neq 0$, then $QP_{11} \neq 0$, ; we shall show that $Q(1_1) \neq 0$. For this, notice first that

$$0 \notin \Delta \Rightarrow ran(1_\Delta(L_{\overline{\alpha}\alpha})) \subset \mathcal{H}_{11} \oplus \mathcal{H}_{10} ;$$

and hence $1_\Delta(L_{\overline{\alpha}\alpha}) = Q$; further, since $1 = 1_0 + 1_1$ is a separating vector for M - see Remark 2.2(d) - and since $1_0 \in \mathcal{H}_{00} \subset ker(L_{\overline{\alpha}\alpha}) \subset ker(1_\Delta(L_{\overline{\alpha}\alpha}))$, we find that

$$Q(1_1) = 1_\Delta(L_{\overline{\alpha}\alpha})(1_1 + 1_0) = 1_\Delta(L_{\overline{\alpha}\alpha})(1) \neq 0 ,$$

and equation (2.8) is verified.

On the other hand, since we clearly have

$$1_\Delta(T) = 1_\Delta(T|_{\mathcal{H}_{10}}) \oplus 1_\Delta(T|_{\mathcal{H}_{11}}) , \tag{2.9}$$

we see that

$$
\begin{aligned}
1_\Delta(A_0) \neq 0 &\Rightarrow 1_{\Delta^2}(T|_{\mathcal{H}_{10}}) \neq 0 \quad \text{(by (2.7))} \\
&\Rightarrow 1_{\Delta^2}(T|_{\mathcal{H}_{11}}) \neq 0 \quad \text{(by (2.9) and (2.8))} \\
&\Rightarrow 1_\Delta(A_1) \neq 0 \quad \text{(by (2.7))} .
\end{aligned}
$$

The reverse implication is proved by exactly analogous reasoning applied with $\alpha\overline{\alpha}$ in place of $\overline{\alpha}\alpha$ (or, by applying the already proved implication to the dual subfactor). $\square$

Proof of Corollary 1.3: The proof of (i) has already been outlined in the paragraph following Definition 1.1, while (ii) is just the verbal translation of the statement that $A_0 \sim_0 A_1$.

For (iii), first observe that if T is a self-adjoint operator, and if $\lambda \in \mathbb{R}$, then $\lambda \in sp_e(T)$ - where we write 'sp_e' for the essential spectrum - if and only if $1_\Delta(T)$ has infinite rank for every open neighbourhood Δ of λ. From this, we see that for any $\lambda \in \mathbb{R}$, the following conditions are equivalent:

(a) $\lambda \in sp_e(T)$ and λ is not an isolated eigenvalue of infinite multipilicity;

(b) every neighbourhood of λ contains infinitely many points of $sp(T)$;

(c) for every open neighbourhood Δ of λ, there exists an infinite collection $\{\Delta_n\}$ of pairwise disjoint open subsets of Δ such that $1_{\Delta_n}(T) \neq 0$ for all n.

It follows that if A, B are self-adjoint operators and if $A \sim_0 B$, and if $\lambda \in \mathbb{R}$, then λ satisfies condition (c) and hence (a) above for A if and only if it satisfies condition (c) and hence (a) for B. □

The fact that the A_p's have the same norm has been known for a while. For related results (such as the fact that they even have the same essential norm), see [P1]. We conclude with a question suggested by Popa, which we formally state in the following 'conjecture'.

Conjecture: Let A denote the adjacency matrix of the principal graph of a finite index subfactor. Then, $sp(A) = sp_e(A)$ in the infinite-depth case.

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