Solutions to Home-work 5

1. If $dim(V) = n$, then the assumption $dim(W_1) + dim(W_2) > n$ is seen to imply that $dim(W_1 \cap W_2) = dim(W_1) + dim(W_2) - dim(W_1 + W_2) > 0$ (since $W_1 + W_2$ being a subspace of V can have dimension at most n), and hence $W_1 \cap W_2 \neq \{0\}$.

   If W_1, W_2 are planes through the origin in $\mathbb{R}^3$, they are two-dimensional subspaces; if they are distinct subspaces, then we may conclude from the last paragraph that $dim(W_1 \cap W_2) > 0$. On the other hand, if $\{u, v\}$ is a basis for W_1 and if $w \in W_2 \backslash W_1$, it is easy to see that $\{u, v, w\}$ must be linearly independent and hence a basis for $\mathbb{R}^3$. We may conclude that $W_1 + W_2 = \mathbb{R}^3$, and that $W_1 \cap W_2$ is one-dimensional. If $\{v\}$ is a basis for $W_1 \cap W_2$, we find that $W_1 \cap W_2 = \mathbb{R}v = \{\alpha v : \alpha \in \mathbb{R}\}$ is the line through the origin, consisting of multiples of v.

2. Let us prove the following

   Assertion: The following conditions on subspaces W_1, W_2 of a finite dimensional vector space V are equivalent:

   (a) $dim(W_1) + dim(W_2) = dim(V)$ and $W_1 \cap W_2 = \{0\}$

   (b) If B_i is any basis for W_i, for $i = 1, 2$, then $B_1 \cup B_2$ is a basis for V.

   (c) Every vector $v \in V$ is uniquely expressible as $v = w_1 + w_2$ with $w_i \in W_i, i = 1, 2$.

   Proof of assertion: $(a) \Rightarrow (c)$: If B_i is a basis for W_i (so $dim(W_i) = |B_i|$), we need to show that $B = B_1 \cup B_2$ is a basis for V. First, notice that $dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2) = dim(V)$ so we must have $V = W_1 + W_2$, so every $v \in V$ is indeed expressible as $v = w_1 + w_2$ with $w_i \in W_i, i = 1, 2$. If also $v = \tilde{w}_1 + \tilde{w}_2$ with $\tilde{w}_i \in W_i, i = 1, 2$, then we must have $w_1 - \tilde{w}_1 = \tilde{w}_2 - w_2 \in W_1 \cap W_2 = \{0\}$ and hence $w_i = \tilde{w}_i, i = 1, 2$ and such a decomposition is unique.

   $(c) \Rightarrow (b)$ is a consequence of the fact that each $w_i \in W_i$ is uniquely expressible as a linear combination of vectors in B_i, for both $i = 1, 2$.

   $(b) \Rightarrow (c)$ and $(c) \Rightarrow (a)$ are obvious.

3. This problem is a routine vereification which may be safely left to the reader.

4. (a) If $x_i \in W_i, \alpha_i \in \mathbb{R}, i = 1, 2$, then $\alpha_1 x_1 + \alpha_2 x_2 \in W_i$ since the subspace W_i is closed under forming linear combinations, for each $i = 1, 2$, and thus $\alpha_1 x_1 + \alpha_2 x_2 \in W_1 \cap W_2$. As for the second assertion, the definition of 'intersection' shows that in fact $W_1 \cap W_2$ contains any *subset* which is contained in both $W_i, i = 1, 2$.

   (b) If $x^{(i)} = w_1^{(i)} + w_2^{(i)} \in W_1 + W_2$, with $w_j^{(i)} \in W_j$, and if $\alpha_i \in \mathbb{R}$ for $i, j = 1, 2$, then observe that

$$\begin{aligned} \sum_{i=1}^{2} \alpha_i x^{(i)} &= \sum_{i=1}^{2} \alpha_i \sum_{j=1}^{2} w_j^{(i)} \\ &= \sum_{j=1}^{2} \sum_{i=1}^{2} \alpha_i w_j^{(i)} \\ &= \sum_{i=1}^{2} \alpha_i w_1^{(i)} + \sum_{i=1}^{2} \alpha_i w_2^{(i)} \\ &\in W_1 + W_2 \ , \end{aligned}$$

thus verifying that $W_1 + W_2$ is closed under forming linear combinations and is hence a subspace. The second assertion is obvious.