Solutions to Home-work 4

1. $(a) \Rightarrow (b)$: Let S be a linearly independent set in V. By hypothesis, there exists a basis B for V with n elements, and also $|S| = n$. By the lemma proved in last class, (since B spans V), there exists a decomposition $B = B_1 \coprod B_2$ of B with $|B_1| = |S|$ such that $S \coprod B_2$ spans V. The hypotheses imply that $|B_1| = |B|$ so $B_2 = \emptyset$, and hence $S = S \coprod \emptyset$ is a spanning set for V. Thus $sp(S) = V$, as desired.

   $(b) \Rightarrow (c)$: If S is a spanning set for V, with $|S| = n$, and if B is any basis for V, then by the lemma alluded to in the last paragraph, there is a decomp[osition $S = S_1 \coprod S_2$ such that $n = |B| = |S_1|$ and $B \coprod S_2$ is a spanning set for V. This means that $S_2 = \emptyset$ so that $S = B$ is a basis.

   $(c) \Rightarrow (a)$: Obvious.

2. Suppose If $(a, b, c) = \alpha(1, 1, 1) + \beta(0, 1, 1) + \gamma(0, 0, 1)$, then it is easy to see that $a = \alpha, \beta = b - \beta = b - a$ and similarly that $\gamma = c - a - b$. Since (a, b, c) is arbitrary, we see that the given set of three vectors does indeed span $\mathbb{R}^3$, and hence must be a basis for $\mathbb{R}^3$, by the previous problem.

3. We leave it as an exercise to the reader to verify that

   $$\{(1, 1, 1, 1), (1, 1, -1, -1), (1, -1, 1, -1), (1, -1, -1, 1)\}$$

   is a linearly independent (or spanning) set, and hence a basis for $\mathbb{R}^4$.

4. By the remark in the solution to 3(c) of HW-3, the set $\{(1, 2), (3, 6)\}$ is linearly dependent, while $\{(1, 2), (3, 8)\}$ is linearly independent, and hence a basis for $\mathbb{R}^2$.

5. If $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in W, \alpha, \beta \in \mathbb{R}$ and $z = (z_1, z_2, z_3) = \alpha x + \beta y$, then

   $$\begin{aligned} 2z_1 + z_2 - z_3 &= 2(\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) - (\alpha x_3 + \beta y_3) \\ &= \alpha(2x_1 + x_2 - x_3) + \beta(2y_1 + y_2 - y_3) \\ &= 0 \end{aligned}$$

   so indeed also $z \in W$ and W is a subspace of $\mathbb{R}^3$. Hence $dim(W) \leq 3$. On the other hand, it is easy to see that $\{(1, 0, 2), (1, 0, 1)\}$ is a linarly independent set in W so that $dim(W) \geq 2$. So indeed W is a two-dimensional (subspace of $\mathbb{R}^3$ and hence a) vector space.

6. For the first assertion, we may clearly assume $V \neq \{0\}$ Suppose V admits a finite spanning set S (which is not $\{0\}$). We have seen (in problem 5 of HW-3) that then S contains a subset which is a minimal spanning set, hence a basis (call it B), for V. Let $|B| = dim(V) = n$. We shall now prove the following:

   Assertion If L is any linearly independent set in V, then there exists a basis which contains L.

   We prove this assertion by induction on $n - |L|$. First suppose $|L| = n$. Then if $v \in V\text{Ł}$, $|L \cup \{v\}| > n$, and hence $L \cup \{v\}$ cannot be linearly independent, since any linearly independent set in an n-dimensional vector space can have at most n elements. So there must be a non-trivial linear combination of $L \cup \{v\}$ which vanishes. Since L is linearly independent, the coefficient of v in this combiation should be non-zero, or in other words, we must have $v \in \ sp(L)$. Since v was arbitrary, this shows that L spans V and must already be a basis.

   Suppose the result is true for any linearly independent set L_1 with $n - |L_1| < n - |L| (> 0)$. Let $W = sp(L)$. Then $dimW < n$. Pick an y $v \in V \setminus W$. Notice that the set $L_1 = L \cup \{v\}$ is linearly independent. (*Reason:* If $L = \{v_1, \cdots, v_m\}$ and set $v = v_{m+1}$. If $\sum_{i=1}^{m+1} \alpha_i v_i = 0$, the assumption $v_{m+1} \notin W$ implies that $\alpha_{m+1} = 0$ (since $L \subset W$). The linear independence of L next forces $\alpha_i = 0 \ \forall i \leq m$.) Since $n - |L_1| = n - (|L| + 1) < n - |L|$, it follows from the induction hypothesis that L_1 - and hence also L - can be extended to a basis, as desired.

   Now if W is any subspace of V, by what we have just shown, W has a basis, say L. Since L is a linearly independent set (in W as well as in V, this L is a subset of a basis, say B, for V; deduce that $dim(W) = |L| \leq |B| = dim(V)$.

7. Since any (finite) spanning set for V can be shrunk to a basis, and since every linearly independent set in a (finite-dimensional vector space) V can be expanded to a basis for V, it is clear that if S, B and L denote an arbitrary spanning set, basis and linearly independent set for V, then

$$|L| \leq |B| \leq |S|.$$

   So if there exists an S with $|S| = n$, then we must have $|L| \leq n < n + 1$, as asserted. The final assertion follows easily. One consequence of this is, for instance, that the vector space of all polynomials is not finite-dimensional.