Solutions to Home-work 2

1.

$$\begin{aligned}(u+(y+(v+x))) &= (u+((v+x)+y)) \quad \text{(by commutativity)}\\ &= (u+(v+(x+y))) \quad \text{(by associativity)}\\ &= (u+v)+(x+y))) \quad \text{(by associativity)}\ .\end{aligned}$$

2.

$$\begin{aligned}u+v=v+w &\Rightarrow (-v)+(u+v)=(-v)+(v+w)\\ &\Rightarrow (-v)+(v+u)=(-v)+(v+w)\\ &\Rightarrow ((-v)+v)+u=((-v)+v)+w\\ &\Rightarrow 0+u=0+w\\ &\Rightarrow u=w\end{aligned}$$

In partricular, if $\tilde{v}$ is a potential $-v$, then $\tilde{v}+v=0=(-v)+v$, which implies $\tilde{v}=-v$.

A consequence of the lasat sentence is that $(-(-v))=v$.

Also $(-1)v+v=((-1)+1)v=0$, and the previous lines then force $(-1)v=-v$.

3. First note that $-(v-w)=(-1)(v+(-1)w)=(-1)v+(-1)(-1)w=-v+(-(-w))=-v+w$ and add u to both sides.

4. $\oplus$ is neither commutative nor associative as shown by $1\oplus 2=1-2=-1\neq 1=2-1=2\oplus 1$ and $(0\oplus 1)\oplus 2=(0-1)-2=-3$ while $0\oplus(1\oplus 2)=0-(1-2)=1$. However, the usual 0 also behaves as a zero for $\oplus$ since $u\oplus 0=u-0=u$, while every v is a candidate for $\ominus v$.

   Scalar multiplication is not associative since $\alpha*(\beta*v)=(-\alpha)(-\beta v)=\alpha\beta v\neq-(\alpha\beta)v=(\alpha\beta)*v$ if $\alpha\beta\neq 0$. $1*v=-v\neq v$ unless $v=0$.

5. (a) The proofs of (a) and (c) are straight-forward.

   (b) The verification of the vector space axioms in $\mathbb{R}[t]$ is an easy consequence of (a) above and the fact that for each $n\geq 0$, the equation

$$T(a_0,a_1\cdots,a_n)=a_0+a_1t+\cdots+a_nt^n$$

defines a bijective mapping T from $\mathbb{R}^n$ to the set $\mathcal{P}_n = \sum_{i=0}^{n} a_i t^i : a_i \in \mathbb{R}\ \forall 1 \leq i \leq n\}$ of polynomials of degree at most n, which *respects all the vector operations - i.e., vector addition and scalar multiplication.*