Solutions to Home-work 11

1. (a) Consider the matrix B whose rows are given by

$$B^r = \begin{cases} A^r & \text{if } r \neq i \\ A^k & \text{if } r = i \end{cases}$$

Since A and B agree except at the i-th row, it follows that $A^i_j = B^i_j$ for all j. However B has two identical rows and consequently has (row-)rank at most $n-1$; in particular B is not invertible and $det(B) = 0$. Appealing now to equation (0.1) of this exercise, we see that

$$\begin{aligned} 0 &= det(B) \\ &= \sum_{j=1}^{n} (-1)^{i+j} b^i_j det(B^i_j) \\ &= \sum_{j=1}^{n} (-1)^{i+j} a^k_j det(A^i_j) \ , \\ &= \sum_{j=1}^{n} a^k_j c^j_i \end{aligned}$$

as desired.

(b) It now follows that

$$\begin{aligned} (AC(A))^k_i &= \sum_{j=1}^{n} a^k_j c^j_i \\ &= \begin{cases} det(A) & \text{if } k = i \text{ by equation (0.1)} \\ 0 & \text{if } k \neq i \text{ by part (a) above} \end{cases} \\ &= (det(A) I_n)^k_i \ . \end{aligned}$$

(c) If $det(A) \neq 0$, it follows from (b) above that A (regarded as an operator on $\mathbb{R}^n$) is onto, hence also 1-1, and invertible; further $(det(A))^{-1}C(A)$ must be the (right-, and hence) inverse of A.

2. Let $E = ((e^i_j))$ be an elementary matrix. We shall illustrate each case with an example. First recall that if

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} ,$$

then $det(A) = aek - ahf + bfg - bdk + cdh - ceg$.

Case (i): Suppose

$$E_1 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

Then

$$E_1 A = \begin{bmatrix} a+3g & b+3h & c+3k \\ d & e & f \\ g & h & k \end{bmatrix} ,$$

and so

$$\begin{aligned} det(E_1 A) &= (a+3g)ek - (a+3g)hf + (b+3h)fg - (b+3h)dk + (c+3k)dh - (c+3k)eg \\ &= det(A) + det\left(\begin{bmatrix} 3g & 3h & 3k \\ b & c & d \\ g & h & k \end{bmatrix}\right) \\ &= det(A) + 0 \\ &= det(A) \ . \end{aligned}$$

The above equation, when applied to $A = I_3$, yields $det(E_1) = 1$ and so, indeed, we have $det(E_1 A) = det(E_1)det(A)$.

Case (ii): Suppose

$$E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} .$$

Then, we find that

$$E_2 A = \begin{bmatrix} g & h & k \\ d & e & f \\ a & b & c \end{bmatrix} ,$$

so that

$$\begin{aligned} det(E_2 A) &= gec - gfb + hfa - hdc + kdb - kea \\ &= -det(A)!. \end{aligned}$$

The above equation, when applied to $A = I_3$, yields $det(E_2) = -1$ and so, indeed, we have $det(E_2 A) = det(E_2)det(A)$.

Case (iii): Suppose

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} .$$

Then

$$E_3A = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5k \end{bmatrix} ,$$

and so

$$\begin{aligned} det(E_3A) &= ae5k - a5hf + bf5g - bd5k + cd5h - ce5g \\ &= 5det(A) . \end{aligned}$$

The above equation, when applied to $A = I_3$, yields $det(E_3) = 5$ and so, indeed, we have $det(E_3A) = det(E_3)det(A)$.

3. The case of 3×3 matrices has already been demonstrated in problem 2. For a 4×4 matrix $A = ((a^i_j))$, say, we find on expanding along the first row that

$$det(A) = \sum_{j=1}^{4} (-1)^{j+1} a^1_j det(A^1_j)$$

expresses $det(A)$ as a sum of four terms each of which is a multiple of the determinant of a 3×3 matrix. Since each $det(A^1_j)$ is the asum of six terms, we see that $det(A)$ is indeed the a sum of $4 \times 6 = 24$ terms.

Finally, a minor extrapolation of these ideas will suggest that the 5×5 case will involve $5 \times 24 = 120$ terms; and by induction, that the general $n \times n$ case will involve $n \times (n-1)! = n!$ terms.