The Riesz representation theorem

V.S. Sunder
Institute of Mathematical Sciences
Madras 600113
INDIA

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1 Introduction

This paper is possibly of only pedagogical interest. While giving a course on measure theory, the author worked out this (fairly elementary) proof of the Riesz Representation Theorem [Rie]. He subsequently learnt that V.S. Varadarajan [VSV] has also given an elementary proof, which uses more or less the same tools; unfortunately, however, back-volumes (as far back as 1959) of that journal are not easily available in India. Even Varadarajan did not seem to have copies of that reprint.

Finally, this author has received enough favourable response to talks on this proof as well as requests for preprints that it seemed plausible that there might be a case for publishing this proof in an Indian journal.

The author would like to record his gratitude to S.M. Srivastava and M.G. Nadkarni for filling in some arguments in the proofs of Proposition 2.3 and the reduction from the locally compact to the compact case, respectively.

A final mathematical note: we restrict ourselves to proving the version of the Riesz Representation Theorem which asserts that 'positive linear functionals come from measures'. Thus, what we call the Riesz Representation Theorem is stated in three parts - as Theorems 2.1, 3.3 and 4.1 - corresponding to the compact metric, compact Hausdorff, and locally compact Hausdorff cases of the theorem.

2 The compact metric case

In this section we shall prove a special and probably the most important case of the theorem - i.e., when the underlying space X is a compact metric space, which is our standing assumption throughout this section.

In this section, the symbol $\mathcal{B}_X$ will denote the *Borel* σ*-algebra of* X; i.e., $\mathcal{B}_X$ is the smallest σ-algebra of subsets of X which contains all compact sets.

Thus, our aim in this section is to prove the following fact.

THEOREM 2.1 *If* $\tau : C(X) \to \mathbb{C}$ *is a linear functional which is positive in the sense of assuming non-negative real values on* *non-negative real-valued continuous functions, then there exists a unique finite positive measure μ defined on the Borel σ-algebra $\mathcal{B}_X$ such that*

$$\tau(f) = \int f d\mu \ .$$

We prepare the way for proving this result, with a few simple results.

LEMMA 2.2 *Let X be a compact Hausdorff space. Then the following conditions on a linear functional $\tau : C(X) \to \mathbb{C}$ are equivalent:*

(a) τ is positive in the sense of the statement of the previous theorem;

(b) τ is a bounded linear functional and $||\tau|| = \tau(1)$, where we write 1 for the constant function identically equal to one.

Proof: This statement is known - see [Arv] for instance - to hold in the more general context where $C(X)$ is replaced by a general (not necessarily commutative) unital C^*-algebra. In the interest of a general readership, we present the proof in the special (commutative) case stated in this lemma.

$(a) \Rightarrow (b)$: Positivity of τ implies that the equation

$$(f, g) = \tau(f\bar{g})$$

defines a 'semi-inner product' on $C(X)$; i.e., the expression (f, g) is sesquilinear in its arguments (meaning $(\sum_{i=1}^{2} \alpha_i f_i, \sum_{j=1}^{2} \beta_j g_j) = \sum_{i,j=1}^{2} \alpha_i \bar{\beta}_j (f_i, g_j)$) and positive semi-definite (meaning $(f, f) \geq 0$ for all f). It is a standard fact - found in any text in functional analysis, such as [Sim] or [Sun] - that every sesquilinear positive-semidefinite form satisfies the celebrated *Cauchy Schwartz inequality*: i.e.,

$$|(f, g)|^2 \leq (f, f)(g, g) \ \forall \ f, g \in C(X) \ .$$

In other words

$$|\tau(f\bar{g})|^2 \leq \tau(|f|^2)\tau(|g|^2) \ \forall \ f, g \in C(X) \ . \tag{2.1}$$

In particular, with $g = 1$, we find that

$$|\tau(f)|^2 \leq \tau(|f|^2)\tau(1).$$

But it follows from positivity that

$$\tau(|f|^2) \leq \tau(||f||^2\ 1) = ||f||^2\tau(1).$$

Thus, we find that

$$|\tau(f)|^2 \leq ||f||^2\tau(1)^2\ \forall\ f;$$

i.e., $||\tau||_{C(X)^*} \leq \tau(1)$. The reverse inequality is obviously valid since $||1|| = 1$; the proof of $(a) \Rightarrow (b)$ is complete.

$(b) \Rightarrow (a)$ We may assume, without loss of generality, that $||\tau|| = \tau(1) = 1$. It will clearly suffice to show that

$$0 \leq f \leq 1 \Rightarrow 0 \leq \tau(f) \leq 1\ .$$

Suppose, to the contrary, that $\tau(f) = z \in \mathbb{C} \setminus [0,1]$ for some $0 \leq f \leq 1$. Then we can find an open disc - with centre z_0 and radius $r > 0$, say - which contains [0,1] but not z. Then, for any $x \in X$, we have $|f(x) - z_0| < r$, and consequently $||f - z_0 1|| < r$; hence $|z - z_0| = |\tau(f - z_0 1)| \leq ||f - z_0 1|| < r$. This contradiction to our assumption that $|z - z_0| \geq r$ completes the proof. □

PROPOSITION 2.3 *Every compact metric space is a continuous image of the (compact metric) space* $2^{\mathbb{N}}$*, the Cartesian product of countably infinitely many copies of a two point space.*

Proof: The proof relies on two facts:

(i) a compact metric space is totally bounded, i,e,. for any $\epsilon > 0$ it is possible to cover X by finitely many sets with diamater less than ϵ; and

(ii) Cantor's intersection theorem which states that the intersection, in a compact metric space, of a decreasing sequence of closed sets with diameters converging to zero is a singleton set.

Indeed, by (i) above, we may find closed sets $\{B_k : 1 \leq k \leq n_1\}$ each with diameter at most one, whose union is X.

Next, since B_j is compact, we can find closed sets $\{B_{j,k} : 1 \leq k \leq m'_j\}$ of diameter at most $\frac{1}{2}$ whose union is B_j; choose $n_2 = max_{1 \leq j \leq n_1}\ m'_j$, and define $B_{j,k} = B_{j,m'_j}$ for $m'_j \leq k \leq n_2$. In other words, we may assume that in the labelling of $B_{j,k}$, the range of the second index k is over the finite index set $\{1, \cdots, n_2\}$ and independent of the first index j. Using (i) repeatedly, an easy induction argument shows that we may, for each q, find a positive integer n_q and closed sets $\{B_{j_1,j_2,\cdots,j_q} : 1 \leq j_i \leq n_i\}$ of diameter at most $\frac{1}{q}$ such that (a) $X = \cup_{j=1}^{n_1} B_j$, and (b) $B_{j_1,j_2,\cdots,j_{q-1}} = \cup_{j_q=1}^{n_q} B_{j_1,j_2,\cdots,j_q}$ for each $j_1, j_2, \cdots, j_{q-1}$.

If $\mathbf{j} = (j_1, j_2, \cdots, j_q, \cdots)$, appeal to (ii) above to find that $\cap_{q=1}^{\infty} B_{j_1,j_2,\cdots,j_q} = \{f(\mathbf{j})\}$ for a uniquely determined function $f : \prod_{q=1}^{\infty}\{1, 2, \cdots, n_q\} \to X$. The hypotheses ensure that if $\mathbf{i}$ and $\mathbf{j}$ agree in the first q co-ordinates, then $f(\mathbf{i})$ and $f(\mathbf{j})$ are at a distance of at most $\frac{2}{q}$ from one another. This shows that the function f is continuous. Finally (a) and (b) of the last paragraph ensure that the function f maps onto X.

Finally, we may clearly assume, without loss of generality, that $n_q = 2^{m_q}$. Then it is clear that there is a (continuous) surjection from $2^{\mathbb{N}}$ to $\prod_{q=1}^{\infty}\{1, 2, \cdots, 2^{m_q}\}$. Combining these two maps we get the desired surjection to X. □

Proof of Theorem 2.1

In addition to the above facts, the proof uses just one more fact, the Hahn-Banach Theorem, which says that any bounded linear functional τ_0 on a subspace V_0 of a normed space V may be extended to a bounded linear functional τ on V such that $||\tau||_{V^*} = ||\tau_0||_{V_0^*}$. This may be found in any text on functional analysis. (See [Sim] or [Sun], for instance.)

The first step of the proof is to observe that the special case of Theorem 2.1, when $X = 2^{\mathbb{N}}$, is a consequence of Caratheodory's Extension Theorem. Indeed, suppose τ is a positive linear functional on $C(2^{\mathbb{N}})$, which we may assume is normalised so that $\tau(1) = 1$. Let

$$\pi_{n]}(j_1, j_2 \cdots, j_n, \cdots) = (j_1, j_2 \cdots, j_n)$$

denote the projection $\pi_{n]} : 2^{\mathbb{N}} \to 2^n$ onto the first n co-ordinates (where we have written 2^n to denote the product of n copies of the two point space). Let $\mathcal{A} = \{\pi_{n]}^{-1}(E) : E \subset 2^n, n \in \mathbb{N}\}$.

Then, $\mathcal{A}$ is a base for the topology of $2^{\mathbb{N}}$, and all the members of $\mathcal{A}$ are open and compact; and $\mathcal{A}$ is an algebra of sets which generates the Borel σ-algebra $\mathcal{B}_{2^{\mathbb{N}}}$. In particular the functions 1_A are *continuous* for each $A \in \mathcal{A}$. Hence, we may define

$$\mu(A) = \tau(1_A) \ \forall \ A \in \mathcal{A} \ . \tag{2.2}$$

The linearity of τ clearly implies that μ is a finitely additive set function on $\mathcal{A}$. Since members of $\mathcal{A}$ are compact and open, no element of $\mathcal{A}$ can be expressed as a countable union of pairwise disjoint non-empty members of $\mathcal{A}$. In other words, we see that any finitely additive set function on $\mathcal{A}$ is automatically countably additive, and hence, by Caratheodory's extension theorem, extends to a probability measure μ defined on all of $\mathcal{B}_X$. If we define $\tau_\mu(f) = \int f d\mu$, then we see from equation (2.2) that τ and τ_μ agree on any continuous function f which factors through some $\pi_{n]}$; since such functions are dense in $C(2^{\mathbb{N}})$, we may conclude that $\tau = \tau_\mu$. In fact, equation (2.2) determines μ uniquely, since a finite measure on $\mathcal{B}$ is uniquely determined by its restriction to any 'algebra of sets' which generates $\mathcal{B}$ as a σ-algebra. In other words, Theorem 2.1 is indeed true when $X = 2^{\mathbb{N}}$

Suppose now that X is a general compact metric space and that τ is a positive linear functional on $C(X)$. We may assume that $||\tau|| = \tau(1) = 1$. Proposition 2.3 guarantees the existence of a continuous surjection $p : 2^{\mathbb{N}} \to X$. Then it is easy to see that the map $p^* : C(X) \to C(2^{\mathbb{N}})$ defined by $p^*(f) = f \circ p$ is an isometric positivity preserving homomorphism of algebras. In particular, we may regard $C(X)$ as a subspace of $C(2^{\mathbb{N}})$ via p^*. It follows from the Hahn-Banach theorem that there exists a bounded linear functional $\tilde{\tau}$ on $C(2^{\mathbb{N}})$ such that

$$\tilde{\tau}(p^*(f)) = \tau(f) \ \forall \ f \in C(X) \tag{2.3}$$

and

$$||\tilde{\tau}|| = ||\tau|| = \tau(1) = \tilde{\tau}(p^*(1)) = \tilde{\tau}(1). \tag{2.4}$$

Deduce now from Lemma 2.2 that $\tilde{\tau}$ is a positive linear functional on $C(2^{\mathbb{N}})$. Conclude from the already proved special case of the theorem for $2^{\mathbb{N}}$ that there exists a positive (in fact probability) measure ν defined on $\mathcal{B}_{2^{\mathbb{N}}}$ such that $\tilde{\tau}(g) = \int_{2^{\mathbb{N}}} g d\nu$.

hence, we see from the 'change of variable formula' that

$$\begin{aligned}\tau(f) &= \tilde{\tau}(p^*(f)) \\ &= \int_{2^{\mathbb{N}}} p^*(f)\ d\nu \\ &= \int_{2^{\mathbb{N}}} f \circ p\ d\nu \\ &= \int_X f\ d(\nu \circ p^{-1})\ ;\end{aligned}$$

hence with $\mu = \nu \circ p^{-1}$, we see that τ is indeed given by integration against μ.

To complete the proof, we only need to establish uniqueness, For this, we first assert that if τ is given by integration against μ, then

$$\mu(K) = \inf\{\tau(f) : 1_K \leq f \in C(X)\} \tag{2.5}$$

for every compact $K \subset X$.

Since

$$1_K \leq f \Rightarrow \mu(K) = \int 1_K d\mu \leq \int f d\mu = \tau(f)\ ,$$

it is clear that $\mu(K)$ is no greater than the infimum displayed in equation (2.5).

Conversely, since compact subsets of X are G_δ sets, we can find a decreasing sequence of open sets U_n such that $K = \cap U_n$. (For instance, we can choose $U_n = \{x \in X : d(x, K) < \frac{1}{n}\}$.) We can find continuous $f_n : X \rightarrow [0, 1]$ such that $1_K \leq f \leq 1_{U_n}$. Then it is clear that $f_n(x) \rightarrow 1_K(x)\ \forall x \in X$; and since $0 \leq f_n \leq 1\ \forall n$, it follows from the dominated convergence theorem that

$$\mu(K) = \int 1_K d\mu = \lim \int f_n d\mu$$

and in particular, $\mu(K)$ is no smaller than the infimum displayed in equation (2.5).

Finally, since any finite measure on a compact metric space is determined by its values on compact sets (see Proposition 4.2), we see that τ indeed determines μ uniquely.

□

3 The general compact Hausdorff case

We assume in this section that X is any compact Hausdorff space. We begin with a couple of elementary lemmas.

LEMMA 3.1 *(a) The following conditions on a closed subset A of X are equivalent:*

(i) A is a G_δ set - i.e., there exists a sequence $\{U_n\}$ of open sets such that $A = \cap_n U_n$;

(ii) there exists a continuous function $f : X \to [0,1]$ such that $A = f^{-1}(\{0\})$.

Proof: $(i) \Rightarrow (ii)$: Compact Hausdorff spaces are normal; so we can, by Urysohn's theorem, find a continuous function $f_n : X \to [0,1]$ such that

$$f_n(x) = \begin{cases} 0 & if\ x \in A \\ 1 & if\ x \notin U_n \end{cases} \quad ;$$

now set $f = \sum_{n=1}^{\infty} 2^{-n} f_n$.

$(ii) \Rightarrow (i)$: Let $U_n = \{x \in X : |f(x)| < \frac{1}{n}\}$. □

In this section, the symbol $\mathcal{B}_X$ will denote the the smallest σ-algebra of subsets of X which contains all the compact G_δ sets. A member of the σ-algebra $\mathcal{B}_X$ is called a *Baire set.* (Note that when X is a compact metric space, there is no distinction between Borel sets and Baire sets; this is because any closed set A in a metric space is a G_δ, as demonstrated by the function $f(x) = d(x, A)$.)

LEMMA 3.2 *The following conditions on a subset $E \subset X$ are equivalent:*

(i) A is a Baire set;

(ii) there exists a continuous function $F : X \to Y$ from X into a compact metric space Y and a Baire - equivalently Borel - set $E \in \mathcal{B}_Y$ such that $A = F^{-1}(E)$.

In particular, any scalar-valued continuous function on X is 'Baire measurable'. Further, the space Y may, without loss of generality, be taken to be $[0,1]^C$ for some countable set C.

Proof: $(ii) \Rightarrow (i)$: Suppose $F : X \to Y$ is a continuous function from X into a compact metric space Y. Let $\mathcal{C} = \{E \in \mathcal{B}_Y : F^{-1}(E) \textit{ is a Baire set}\}$. First notice that if K is any closed set in Y, then K is also a G_δ set, and consequently $F^{-1}(K)$ is a compact G_δ set and hence a Baire set in X; hence $\mathcal{C}$ contains all closed sets in Y. Since the definition shows that $\mathcal{C}$ is clearly a σ-algebra, it follows that $\mathcal{C} = \mathcal{B}_Y$.

$(i) \Rightarrow (ii)$: Let $\mathcal{B}$ denote the collection of those subsets $A \subset X$ which are inverse images, under continuous functions into Y - where $Y = [0,1]^C$ for some countable set C - of Borel sets in Y. We check now that $\mathcal{B}$ is closed under countable unions. Suppose $A_n = F_n^{-1}(E_n)$ for some $E_n \in \mathcal{B}_{Y_n}$, where $F_n : X \to Y_n = [0,1]^{C_n}$ (with C_n some countable set) is continuous ; we may assume, without loss of generality, that the index sets C_n are pairwise disjoint. Then, let $C = \cup_n C_n$, so we may identify $Y = [0,1]^C$ with $\prod_n Y_n$. Write π_n for the natural projection mapping of Y onto Y_n. Define $F : X \to Y$ by requiring that $\pi_n \circ F = F_n$, and define $E'_n = \pi_n^{-1}(E_n)$. The definitions show that F is continuous, $E'_n \in \mathcal{B}_Y$, so that $\cup_n E'_n \in \mathcal{B}_Y$ and

$$\cup_n A_n = \cup_n F^{-1}(E'_n) = F^{-1}(\cup_n E'_n) \ ;$$

so $\mathcal{B}$ is indeed closed under countable unions. Since it is trivially closed under complementation, it follows that $\mathcal{B}$ is a σ-algebra of sets. So, in order to complete the proof of the lemma, it remains only to prove that $\mathcal{B}$ contains all compact G_δ sets; but this is guaranteed by Lemma 3.1. □

We can now state the Riesz Representation Theorem for general compact Hausdorff spaces.

THEOREM 3.3 *If $\tau : C(X) \to \mathbb{C}$ is a positive linear functional, then there exists a unique finite positive μ defined on the Baire σ-algebra $\mathcal{B}_X$ such that*

$$\tau(f) = \int f d\mu \ .$$

Proof: Let

$$\mathcal{C} = \{(Y, \pi) : \pi \text{ is a continuous map of } X \text{ onto a metric space } Y\} \ .$$

For $(Y, \pi), (Y_1, \pi_1), (Y_2, \pi_2) \in \mathcal{C}$, let us write

$$\begin{aligned} \mathcal{B}_\pi &= \{\pi^{-1}(E) : E \in \mathcal{B}_Y\}, \text{ and} \\ (Y_1, \pi_1) \leq (Y_2, \pi_2) &\Leftrightarrow \exists\, \psi : Y_2 \to Y_1 \text{ such that } \pi_1 = \psi \circ \pi_2 \end{aligned}$$

The proof involves a series of assertions:

(1) If $(Y, \pi) \in \mathcal{C}$, there exists a unique measure μ_Y defined on $\mathcal{B}_Y$ such that

$$\int_Y g \, d\mu_Y = \tau(g \circ \pi), \ \forall\, g \in C(Y) \ . \tag{3.6}$$

(Reason: The equation

$$\tau_Y(g) = \tau(g \circ \pi) \tag{3.7}$$

defines a positive linear functional on $C(Y)$, and we may apply Theorem 2.1 to the compact metric space Y.)

(2) If $(Y_1, \pi_1), (Y_2, \pi_2) \in \mathcal{C}$, and if $(Y_1, \pi_1) \leq (Y_2, \pi_2)$, so that there exists a continuous map $\psi : Y_2 \to Y_1$ such that $\pi_1 = \psi \circ \pi_2$ then $\mu_{Y_1} = \psi_*(\mu_{Y_2})$ (meaning that $\mu_{Y_1}(E) = \mu_{Y_2}(\psi^{-1}(E))$, $\forall\, E \in \mathcal{B}_{Y_1}$.

(*Reason:* For arbitrary $g \in C(Y_1)$, an application of the 'change of variable formula' shows that

$$\begin{aligned} \int_{Y_1} g \, d(\psi_*(\mu_{Y_2})) &= \int_{Y_2} g \circ \psi \, d\mu_{Y_2} \\ &= \tau(g \circ \psi \circ \pi_2) \\ &= \tau(g \circ \pi_1) \ , \end{aligned}$$

and the uniqueness assertion of (1) above establishes the desired equality of measures.)

(3) If $\{(Y_n, \pi_n) : n = 1, 2, \cdots\} \subset \mathcal{C}$, then there exists $(Y, \pi) \in \mathcal{C}$ such that $(Y_n, \pi_n) \leq (Y, \pi)\ \forall n$.

(*Reason:* Define $\pi : X \to \prod_{n=1}^\infty Y_n$ by $\pi(x) = (\pi_1(x), \pi_2(x), \cdots)$, set $Y = \pi(X)$, and let $\psi_n : Y \to Y_j$ be the restriction to Y of the projection onto Y_n.)

Finally, it is a direct consequence of Lemma 3.2 that $\mathcal{B}_X = \cup_{(Y,\pi)\in\mathcal{C}}\mathcal{B}_\pi$. The above assertions show that there exists a unique well-defined and countably additive set-function μ on $\mathcal{B}_X$ with the property that $\mu(\pi^{-1}(E)) = \mu_Y(E)$ whenever $E \in \mathcal{B}_Y, (Y,\pi) \in \mathcal{C}$. In other words, $\pi_*(\mu) = \mu_Y$ so $\int_X g \circ \pi \; d\mu = \int_Y g d\mu_Y = \tau(g \circ \pi) \; \forall \; g \in \; C(Y)$. Now if $f \in C(X)$, set $Y = f(X), \pi = f$, define $g \in C(Y)$ by $g(z) = z \; \forall z \in Y$, and deduce from the previous sentence that indeed

$$\int_X f \; d\mu \;=\; \tau(f) \;,$$

as desired.

As for uniqueness, it is seen, exactly as in the compact metric case, that equation (2.5) is valid for every compact G_δ set K. Then Corollary 4.3 shows that τ determines μ uniquely.

□

4 The locally compact case

In this section, we assume that X_0 is a locally compact Hausdorff space. We shall write

$$C_c(X_0) = \{f : X_0 \to \mathbb{C} : f \text{ continuous, supp}(f) \text{ compact}\} \;,$$

where we write 'supp(f)' to denote the support of f, i.e., the closure of $\{x \in X_0 : f(x) \neq 0\}$. As before, we shall let $\mathcal{B}_{X_0}$ denote the σ-algebra generated by compact G_δ subsets of X_0.

Let us call a positive measure μ defined on $\mathcal{B}_{X_0}$ *inner regular* if: (i) $\mu(K) < \infty$ for every compact G_δ subset K of X_0, and (ii) $\mu(E) = sup\{\mu(K) : K \text{ a compact } G_\delta \text{ subset of } E\}$.

We now wish to derive the following result from the preceding sections.

Theorem 4.1 *If $\tau : C_c(X_0) \to \mathbb{C}$ is a linear functional which is positive - meaning $C_c(X_0) \ni f \geq 0 \Rightarrow \tau(f) \geq 0$ - then there exists a unique positive inner regular measure μ defined on $\mathcal{B}_{X_0}$ such that*

$$\tau(f) = \int f d\mu \; \forall \; f \in C_c(X_0) \;.$$

Proof: Let K be a compact G_δ subset of X_0, and suppose $K = \cap_n U_n$, with U_n open. We may, and do, assume without loss of generality that U_n has compact closure and that $\overline{U_{n+1}} \subset U_n$. For each n, pick a continuous function $\phi_n : X \to [0,1]$ such that $1_{U_{n+1}} \leq \phi_n \leq 1_{U_n}$, i.e.,

$$\phi_n(x) \;=\; \begin{cases} 0 & \text{if } x \notin U_n \\ 1 & \text{if } x \in U_{n+1} \end{cases} \quad .$$

The construction implies that

$$0 \leq \phi_{n+1} = \phi_n \phi_{n+1} \leq \phi_n \ . \tag{4.8}$$

Assertion: Suppose now that $f \in C(K)$ and $f \geq 0$. Suppose[1] $\widetilde{f} \in C_c(X_0)$ is a non-negative extension of f - i.e., $\widetilde{f}|_K = f$. Then $lim_{n\to\infty}\tau(\widetilde{f}\phi_n)$ exists and this limit depends only on f and is independent of the choices of any of $U_n, \phi_n, \widetilde{f}$.

Reason: To start with, the positivity of τ and equation (4.8) imply that $\{\tau(\widetilde{f}\phi_n) : n = 1, 2, \cdots\}$ is a non-increasing sequence of non-negative numbers, and hence converges.

If g is another continuous non-negative function with compact support which extends f, and if $\epsilon > 0$, then $\overline{U_n} \cap \{x \in X_0 : |\widetilde{f}(x) - g(x)| \geq \epsilon\}$ is seen to be a a decreasing sequence of compact sets whose intersection is empty. So, there exists an n such that

$$x \in U_n \Rightarrow |\widetilde{f}(x) - g(x)| < \epsilon \ ;$$

it follows from the positivity of τ that

$$|\tau(\widetilde{f}\phi_k) - \tau(g\phi_k)| \leq \epsilon\tau(\phi_k)$$

for all large k. Since $\lim\ \tau(\phi_n)$ exists (by virtue of the conclusion of the previous paragraph applied to $f = 1_K$, with $\widetilde{f} = 1_{\overline{U_1}}$ (say)) this establishes that the limit of the Assertion is indeed independent of $\widetilde{f}$.

[1]To see that such extensions exist, first choose an open set U and a compact set L such that $K \subset U \subset L$, and appeal to Tietze's extension theorem to find a continuous $g : X \to [0,1]$ with the property that $g|_K = f$ and $g(x) = 0\ \forall x \in L \setminus U$, and finally let $\widetilde{f}$ be g on L and 0 outside L.

Suppose $\{(V_n, \psi_n)\}_n$ is an alternative choice to $\{(U_n, \phi_n)\}_n$ in the sense that (i) V_n is a sequence of open sets such that $K = \cap_n V_n$, (ii) $\psi_n \in C_c(X_0)$, $\psi_n : X_0 \to [0,1]$ and (iii)

$$\psi_n(x) \;=\; \begin{cases} 0 & \text{if } x \notin V_n \\ 1 & \text{if } x \in V_{n+1} \end{cases} \quad .$$

Then, it may be seen, exactly as above, that if $\epsilon > 0$ is arbitrary, then $\sup_{x \in X_0} |\widetilde{f}(x)(\phi_n(x) - \psi_n(x)| < \epsilon$ for large n, and hence deduced from positivity of τ that

$$lim_{n\to\infty}\tau(\widetilde{f}\phi_n) \;=\; lim_{n\to\infty}\tau(\widetilde{f}\psi_n) \;,$$

thereby completing the proof of the assertion.

It is clear that there exists a linear functional τ_K on $C(K)$ such that

$$\tau_K(f) = lim_n \tau(\widetilde{f}\phi_n)$$

for any non-negative f (and $\widetilde{f}$ and ϕ_k's as above). Since τ_K is a positive functional on $C(K)$, we may deduce from Theorem 3.3 that there exists a unique **finite positive** measure μ_K defined on $\mathcal{B}_K$ such that

$$\tau_K(f) = \int_K f \; d\mu_K \;\forall\; f \in C(K) \;.$$

Notice that the collection Λ of compact G_δ sets is a directed set with respect to the ordering defined by

$$K \leq L \;\Leftrightarrow\; K \subset \; Int(L)$$

where $Int(L)$ denotes the interior of L.

We wish now to show that the family $\{\mu_K : K \in \Lambda\}$ is consistent in the sense that

$$K, L \in \Lambda, K \leq L \Rightarrow \;\mu_L|_K = \mu_K. \tag{4.9}$$

First notice that it follows from the definitions that if $g \in C_c(X_0)$ and if $supp(g) \leq L$, then

$$\int_L g \; d\mu_L = \tau(g) \;. \tag{4.10}$$

Suppose now that $K, L \in \Lambda$ and $K \leq L$. We may find a sequence $\{U_n, \phi_n\}$ as above, such that $K = \cap_n U_n$ and $U_1 \subset L$. Now, if $f \in C(K)$ is arbitrary, and if $\widetilde{f}$ is any extension of f to a compactly supported continuous function with $supp(\widetilde{f}) \leq L$, deduce from equation (4.10) and the dominated convergence, for instance, that

$$\begin{aligned}
\int_K f \; d\mu_K &= \lim_n \tau(\widetilde{f}\phi_n) \\
&= \lim_n \int_L (\widetilde{f}\phi_n) \; d\mu_L \\
&= \int 1_K f \; d\mu_L \\
&= \int_K f \; d(\mu_L)|_K \; .
\end{aligned}$$

The uniqueness assertion in Theorem 3.3 then shows that equation (4.9) is indeed valid.

Finally, for any set $E \in \mathcal{B}_{X_0}$, notice that $E \cap K \in \mathcal{B}_K \; \forall K \in \Lambda$ and that $\{\mu_K(E \cap K) : K \in \Lambda\}$ is a non-decreasing net of real numbers which must converge to a number - call it $\mu(E)$ - in $[0, \infty]$. Note that

$$\mu(E) = \sup_{K \in \Lambda} \mu_K(E \cap K) \; \forall E \in \mathcal{B}_{X_0}. \tag{4.11}$$

We wish to say that this μ is the measure on $\mathcal{B}_{X_0}$ whose existence is asserted by the theorem. Suppose $E = \coprod_{n=1}^{\infty} E_n$ where $E, E_n \in \mathcal{B}_{X_0}$. Then

$$\begin{aligned}
\mu(E) &= \sup_{K \in \Lambda} \mu_K(E \cap K) \\
&= \sup_{K \in \Lambda} \mu_K(\coprod_{n=1}^{\infty} E_n \cap K) \\
&= \sup_{K \in \Lambda} \sum_{n=1}^{\infty} \mu_K(E_n \cap K) \\
&= \sup_{K \in \Lambda} \sup_N \sum_{n=1}^{N} \mu_K(E_n \cap K)
\end{aligned}$$

$$
\begin{aligned}
&= \sup_N \sup_{K\in\Lambda} \sum_{n=1}^{N} \mu_K(E_n \cap K) \\
&= \sup_N \lim_{K\in\Lambda} \sum_{n=1}^{N} \mu_K(E_n \cap K) \\
&= \sup_N \sum_{n=1}^{N} \lim_{K\in\Lambda} \mu_K(E_n \cap K) \\
&= \sup_N \sum_{n=1}^{N} \mu(E_n) \\
&= \sum_{n=1}^{\infty} \mu(E_n) \ ,
\end{aligned}
$$

and hence μ is indeed countably additive, i.e., a measure defined on $\mathcal{B}_{X_0}$.

Now, if $f \in C_c(X_0)$, it follows from equations (4.10) and (4.9) that if $L \in \Lambda$ satisfies $supp(f) \leq L$, then

$$\tau(f) \ = \ \int_L f \ d\mu_L \ = \ \int_{X_0} f \ d\mu.$$

Finally, if K is any compact G_δ, then it is clear that

$$\mu(K) \ = \ \mu_K(K) \ < \ \infty \ .$$

Let us defer, for a moment, the proof of inner regularity of our μ, and verify, instead, that there is at most one inner regular measure μ related to a τ as in the theorem. By inner regularity, it suffices to observe that

$$\mu(K) = \inf\{\tau(f) : 1_K \leq f \in C_c(X_0)\}$$

for all compact G_δ subsets K. But this is proved exactly as in the proof of equation (2.5).

Finally, if $E \in \mathcal{B}_{X_0}$ is arbitrary, we have, by definition of our μ, and Corollary 4.3,

$$
\begin{aligned}
\mu(E) &= \sup_{K\in\Lambda} \mu(E \cap K) \\
&= \sup_{K\in\Lambda} \ \sup_{C\subset E\cap K,\ C \ compact \ G_\delta} \mu(C)
\end{aligned}
$$

so indeed

$$\mu(E) = \sup_{C \subset E,\ C\ compact\ G_\delta} \mu(C)$$

The proof of the theorem is complete, modulo that of the following proposition and its corollary. □

PROPOSITION 4.2 *Suppose μ is a finite positive measure defined on the Borel σ-algebra $\mathcal{B}_X$ of a compact metric space. Then, for any Borel set $E \in \mathcal{B}_X$, we have*

$$\begin{aligned}\mu(E) &= \sup\{\mu(K) : K \subset E \text{ and } K \text{ is compact}\} \\ &= \inf\{\mu(U) : E \subset U \text{ and } U \text{ is open}\}.\end{aligned}$$

Proof: Let $\mathcal{M}$ denote the class of all those $E \in \mathcal{B}_X$ which satisfy the conclusion of the Proposition.

Suppose next that K is a compact set in X. Then the first of the desired identities is clearly satisfied by K. On the other hand, we may choose a sequence $\{V_n\}_{n=1}^\infty$ of open sets such that $K = \cap_{n=1}^\infty V_n$. It is then seen - from the fact that finite measures are 'continuous from above' - that if we set $U_n = \cap_{m=1}^n V_m$, then

$$\mu(K) = \lim_{n \to \infty} \mu(U_n) \ ;$$

hence, K also satisfies the second of the desired identities. Thus $\mathcal{M}$ contains all compact sets.

We claim next that $\mathcal{M}$ is an algebra of sets. So suppose $E_i \in \mathcal{M}$ for $i = 1, 2$. Let $\epsilon > 0$. Then we can find compact sets K_1, K_2 and open sets U_1, U_2 such that $K_i \subset E_i \subset U_i$ and $\mu(U_i \setminus K_i) < \epsilon$. Then clearly $K = K_1 \cup K_2$ is a compact subset of $E = E_1 \cup E_2$ and $U = U_1 \cup U_2$ is an open superset of E and we have

$$\begin{aligned}\mu(U \setminus K) &\leq \mu(U_1 \setminus K) + \mu(U_2 \setminus K) \\ &\leq \mu(U_1 \setminus K_1) + \mu(U_2 \setminus K_2) \\ &< 2\epsilon \ ;\end{aligned}$$

the arbitrariness of ϵ shows that indeed $E \in \mathcal{M}$.

Since complements of compact sets in X are open (and conversely), the class $\mathcal{M}$ is seen to be closed under the formation of complements. Hence $\mathcal{M}$ indeed contains the algebra generated by all compact sets in X. Hence, in order to complete the proof, it suffices (by the 'monotone class theorem') to verify that $\mathcal{M}$ is a monotone class of sets.

So suppose $E_n \in \mathcal{M}, n = 1, 2, \cdots$, and let $\epsilon > 0$. Pick compact sets K_n and open sets U_n such that $K_n \subset E_n \subset U_n$ and $\mu(U_n \setminus K_n) < \frac{\epsilon}{2^n}$ for each n.

If $E_n \uparrow E$, pick a large m so that $\mu(E \setminus E_m)) < \epsilon$, and conclude, with $U = \cup_{n=1}^{\infty} U_n$ and $K = K_m$ that U is open, K is compact, $K \subset E \subset U$ and that

$$\begin{aligned} \mu(U \setminus E) &\leq \sum_{n=1}^{\infty} \mu(U_n \setminus E) \\ &\leq \sum_{n=1}^{\infty} \mu(U_n \setminus E_n) \\ &\leq \sum_{n=1}^{\infty} \mu(U_n \setminus K_n) \\ &< \epsilon \ , \end{aligned}$$

while

$$\mu(E \setminus K_m) \leq \mu(E \setminus E_m) + \mu(E_m \setminus K_m) < 2\epsilon \ .$$

Since ϵ was arbitrary, this shows that $E \in \mathcal{M}$.

The case when $E_n \downarrow E$ is similarly seen (take $U = U_k$ and $K = \cap_n K_n$, with k so large that $\mu(E_k \setminus E) < \epsilon$) to imply that $E \in \mathcal{M}$, thereby showing that $\mathcal{M}$ is a monotone class containing the algebra generated by the collection of all compact sets. This completes the proof of the proposition. □

COROLLARY 4.3 *Any finite positive measure deined on the Baire σ-algebra $\mathcal{B}_X$ of a compact Hausdorff space X is inner regular.*

Proof: By Lemma 3.2, if A is any Baire subset of X, then there exists a compact metric space Y, a Borel subset $E \subset Y$ and a continuous map $f : X \to Y$ such that $A = f^{-1}(E)$. Apply Proposition 4.2 to the measure $\nu = \mu \circ f^{-1}$ defined on $\mathcal{B}_Y$, to find compact sets $C_n \subset E$ such that $\nu(E) = \sup_n \nu(C_n)$. Notice that $K_n = f^{-1}(C_n)$ is a closed, hence compact, clearly G_δ subset of X such that $K_n \subset A$ and conclude that

$$\mu(A) = \nu(E) = \sup_n \nu(C_n) = \sup_n \mu(K_n) \ .$$

□

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