$\mathcal{A}$ a C^*-algebra.

Defn :- By a right $\mathcal{A}$-module we shall mean a vector space X together with a bilinear pairing

$$X \times \mathcal{A} \longrightarrow X \quad (x,a) \longmapsto x.a$$

such that $(x.a).b = x.(ab) \quad \forall x \in X,\ a,b \in \mathcal{A}$

$$(\lambda x).a = x.(\lambda a) \quad \forall \lambda \in \mathbb{C},\ x \in X,\ a \in \mathcal{A}.$$

If $\mathcal{A}$ is unital then we do not need this condn.

We write $X_{\mathcal{A}}$ to emphasize X is being viewed as right $\mathcal{A}$-module.

Remark :- Algebraists do not demand X to be a vector space because they deal with rings with identity. So, $\mathcal{A}$ contains a copy of the base field.

Defn :- A right inner product $\mathcal{A}$ module is a right $\mathcal{A}$-module X with a pairing

$\langle \cdot , \cdot \rangle_{\mathcal{A}} : X \times X \longrightarrow \mathcal{A}$ such that

ⓐ $\langle x, \lambda y + \mu z \rangle_{\mathcal{A}} = \lambda \langle x,y \rangle_{\mathcal{A}} + \mu \langle y, z \rangle_{\mathcal{A}}$

ⓑ $\langle x, y.a \rangle_{\mathcal{A}} = \langle x,y \rangle_{\mathcal{A}}\, a \quad \forall x,y \in X,\ a \in \mathcal{A}$

ⓒ $\langle x, y \rangle_{\mathcal{A}}^{*} = \langle y, x \rangle_{\mathcal{A}}$

ⓓ $\langle x, x \rangle_{\mathcal{A}} \geq 0$ as an element of $\mathcal{A}$

ⓔ $\langle x, x \rangle_{\mathcal{A}} = 0 \implies x = 0.$

Remark :- condns ⓐ and ⓒ imply that $\langle \cdot , \cdot \rangle_{\mathcal{A}}$ is conjugate linear in the first variable.

$$\langle \lambda x + \mu y, z \rangle_{\mathcal{A}} = \langle z, \lambda x + \mu y \rangle_{\mathcal{A}}^{*}$$
$$= (\lambda \langle z, x \rangle_{\mathcal{A}} + \mu \langle z, y \rangle_{\mathcal{A}})^{*}$$
$$= \bar{\lambda} \langle x, z \rangle_{\mathcal{A}} + \bar{\mu} \langle y, z \rangle_{\mathcal{A}}.$$

Similarly ⓑ and ⓒ imply that

$$\langle x.a, y \rangle_{\mathcal{A}} = a^{*} \langle x, y \rangle_{\mathcal{A}}$$

It follows that $\operatorname{span}\{\langle x, y \rangle_{\mathcal{A}} \mid x, y \in X\}$ is a two sided

Example :- If we take $\mathcal{A} = \mathbb{C}$, then usual inner product spaces over $\mathbb{C}$ in which the $\mathbb{C}$-valued inner products are conjugate linear in the first variable.

Example :- $X = \mathcal{A}$ with $x.a =$ usual multiplication in the C^{*}-algebra $\mathcal{A}$. and $\langle x, y \rangle_{\mathcal{A}} = x^{*}.y$.

The axioms are easily verified except for (e), which follows from the C^{*}-identity

$$\langle a, a \rangle_{\mathcal{A}} = 0 \iff a^{*}a = 0 \iff \|a\|^2 = \|a^{*}a\| = 0$$
$$\iff a = 0.$$

3. Example :- Let $p \in M_n(\mathcal{A})$ be s.t $p^2 = p = p^*$.
Recall if $A = ((a_{ij}))$ then $(A^*)_{ij} = a^*_{ji}$.

Define $\mathcal{E} = p.\mathcal{A}^n$
Then $\mathcal{E}$ is a right $\mathcal{A}$-module.
The inner product is defined by

$$\langle \underline{x}, \underline{y} \rangle_{\mathcal{A}} = \sum_1^n x_i^* y_i.$$

1. Lemma :- (The Cauchy-Schwarz inequality)
If X is a preinner product $\mathcal{A}$-module (this means ⓐ - ⓓ holds) and if $x, y \in X$
then

$$\text{①} \qquad \langle x, y \rangle_{\mathcal{A}}^* \langle x, y \rangle_{\mathcal{A}} \leqq \| \langle x, x \rangle_{\mathcal{A}} \| \langle y, y \rangle_{\mathcal{A}}$$

as elements of the C^*-alg $\mathcal{A}$. In fact we do not need $\mathcal{A}$ to be a C^*-algebra. Inequality ① holds if X is a right $\mathcal{A}_0$ module for a dense $*$-subalgebra $\mathcal{A}_0$ of a C^*-algebra $\mathcal{A}$ and X has a pairing satisfying ⓐ - ⓓ provided we interpret the inequalities in ⓓ and ①. as holding in the completion $\mathcal{A}$ of $\mathcal{A}_0$.

Remark :- To prove this lemma we need to know that an element a of a C^* algebra is positive if $\rho(a) \geq 0$ $\forall$ state ρ of A. To see this suppose that $\rho(a) \geq 0$ $\forall$ states ρ and choose a faithful representation $\pi : A \to B(H)$. Then $x \mapsto \langle \pi(x)h, h\rangle$ is a state and by our hypothesis $\langle \pi(a)h, h\rangle \geq 0$ $\forall h, \|h\| = 1$.

Thus $\pi(a)$ is a +ve operator in $B(H)$

This means $\sigma_{B(H)}(\pi(a)) \subseteq [0, \infty)$.

By spectral permanence $\sigma_A(a) \subseteq [0, \infty)$.

Hence $a \geq 0$ in A.

Proof of lemma :- It is enough to show that $\rho(\langle x, y\rangle_A^* \langle x, y\rangle_A) \leq \|\langle x, x\rangle_A\| \cdot \rho(\langle y, y\rangle_A)$.

$\forall \rho \in S(A)$ = state space of A.

Fix ρ. Then $(w, z) \mapsto \rho(\langle w, z\rangle_A)$ is a positive semidefinite form on X and the ordinary Cauchy-Schwarz ineq. implies that

$$|\rho(\langle w, z\rangle_A)| \leq \rho(\langle w, w\rangle_A)^{1/2} \rho(\langle z, z\rangle_A)^{1/2}$$

Putting $w = x\langle x, y\rangle$ and $z = y$ we get

$$\rho(\langle x,y\rangle_{\mathcal{A}}^{*}\langle x,y\rangle_{\mathcal{A}}) = \rho(\langle x\langle x,y\rangle_{\mathcal{A}}, y\rangle)$$

$$\leq \rho(\langle x\langle x,y\rangle_{\mathcal{A}}, x\langle x,y\rangle_{\mathcal{A}}\rangle)^{1/2}\rho(\langle y,y\rangle_{\mathcal{A}})^{1/2}$$

$$= \rho(\langle x,y\rangle_{\mathcal{A}}^{*}\langle x,x\rangle_{\mathcal{A}}\langle x,y\rangle_{\mathcal{A}})^{1/2}\rho(\langle y,y\rangle_{\mathcal{A}})^{1/2}$$

We now use $b^{*}cb \leq \|c\| b^{*}b \quad \forall b, \ \forall c \geq 0$

and deduce

(2) $$\rho(\langle x,y\rangle_{\mathcal{A}}^{*}\langle x,y\rangle_{\mathcal{A}}) \leq \|\langle x,x\rangle_{\mathcal{A}}\| \rho(\langle x,y\rangle_{\mathcal{A}}^{*}\langle x,y\rangle_{\mathcal{A}})^{1/2} \times \rho(\langle y,y\rangle_{\mathcal{A}})^{1/2}$$

Squaring and cancelling a factor of (2) we get

$$\rho(\langle x,y\rangle_{\mathcal{A}}^{*}\langle x,y\rangle_{\mathcal{A}}) \leqq \|\langle x,x\rangle_{\mathcal{A}}\| . \langle y,y\rangle_{\mathcal{A}}.$$

2nd proof :- Exercise session

Recall the proof in the pre Hilbert space case :-

Let H be a $\mathbb{C}$-vector space with a nonnegative definite or +ve semidefinite inner product.

Then $$|\langle x,y\rangle| \leqq (\langle x,x\rangle)^{1/2}(\langle y,y\rangle)^{1/2}$$

Case 1 :- Both $\langle x,x\rangle = \langle y,y\rangle = 0$.

Then for $\forall \alpha$, $0 \leq \langle \alpha x+y, \alpha x+y\rangle$

$$= \overline{\alpha}\langle x,y\rangle + \alpha\overline{\langle x,y\rangle}$$

$$= 2\,\mathrm{Re}\, \alpha\overline{\langle x,y\rangle}$$

Taking $\alpha = -\langle x,y\rangle$ we get

$$-2|\langle x,y\rangle|^{2} \geq 0$$

This can happen only if $\langle x,y\rangle = 0$.

Case 2:- At least one of $\langle x,x\rangle$ or $\langle y,y\rangle$ is nonzero. Without loss of generality we can assume $\langle x,x\rangle = 1$. Here the role of x & y is interchangeable and that allows us to make this w.l.g hypothesis.

$$0 \le \langle \alpha x + y, \alpha x + y\rangle$$
$$= \bar{\alpha}\langle x,y\rangle + |\alpha|^2\langle x,x\rangle + \alpha\overline{\langle x,y\rangle} + \langle y,y\rangle$$
$$= 2\,\mathrm{Re}\,\alpha\overline{\langle x,y\rangle} + |\alpha|^2 + \langle y,y\rangle$$

put $\alpha = -\langle x,y\rangle$ to conclude

$$0 \le |\langle x,y\rangle|^2 - 2|\langle x,y\rangle|^2 + \langle y,y\rangle$$

or, $$|\langle x,y\rangle| \le (\langle y,y\rangle)^{1/2}\cdot(\langle x,x\rangle)^{1/2}.$$

Now we do the case of Hilbert C^*-modules :-

Case 1:- $\langle x,x\rangle_A = \langle y,y\rangle_A = 0$.

Then $\forall a \in A$,

$$(3)\quad 0 \le \langle x.a+y, x.a+y\rangle_A$$
$$= a^*\langle x,x\rangle_A a + \langle y,y\rangle_A + a^*\langle x,y\rangle_A + \langle x,y\rangle_A^*.a$$

Put $a = -\langle x,y\rangle$ to obtain,

$$0 \le -2\langle x,y\rangle^*\langle x,y\rangle$$

This can happen only if $\langle x,y\rangle = 0$

So, in this case (1) holds.

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Case 2:- $\langle x,x\rangle \neq 0$ w.l.g we can assume $\|\langle x,x\rangle\| = 1$.

From ineq. (3) we obtain,

$$0 \leq a^* a + \langle y,y\rangle_A + a^*\langle x,y\rangle_A + \langle x,y\rangle_A^* a.$$

(Note we have used
$$a^*\langle x,x\rangle_A a \leq a^* a . \|\langle x,x\rangle_A\|)$$

As before we put $a = -\langle x,y\rangle$ to obtain

$$\langle x,y\rangle_A^* \langle x,y\rangle_A \leq \langle y,y\rangle_A.$$

Case 3:- $\langle y,y\rangle_A \neq 0$, $\langle x,x\rangle_A = 0$.

Then by case 2 we have

$$0 \leq \langle y,x\rangle_A^* \langle y,x\rangle_A \leq \|\langle y,y\rangle_A\| . \langle x,x\rangle_A = 0$$

Therefore, $\langle y,x\rangle = 0$ implying $\langle x,y\rangle = 0$.

Thus even in this case we get (1).

Cor:- If the innerproduct satisfies (a) - (d) then $N = \{x \in X \mid \langle x,x\rangle_A = 0\}$ is a right pre/semi innerproduct A-module.

Pf:- We need to show $x \in N$, $a \in A$ imply $x.a \in N$. But that is obvious because $\langle x.a, x.a\rangle_A = a^* \langle x,x\rangle_A a = 0$.

To show N is closed under addition let $x, y \in N$

$\langle x+y, x+y\rangle_A = \langle x,y\rangle_A + \langle y,x\rangle_A = 0.$

Because by CS inear

$0 \leq \langle x,y\rangle_A^* \langle x,y\rangle_A \leq 0 \Rightarrow \langle x,y\rangle_A = 0.$

Similarly $\langle y,x\rangle_A = 0.$

Cor:- On X/N, $\langle x+N, y+N\rangle_A = \langle x,y\rangle_A$ is a well defined map and this makes X/N into a right inner product A-module.

Proof:- To show well definedness we

$y \in N \Rightarrow \langle x,y\rangle_A = 0.$

That follows from

$$0 \leq \langle y,x\rangle_A^* \langle x,y\rangle_A \leq \|\langle y,y\rangle_A\| \langle x,x\rangle = 0$$

$\therefore \langle x,y\rangle_A = 0.$

Remark :- The inner product on the right-A module X/N satifeis ⓐ - (e).

Cor:- Let X be an ~~right~~ inner product right A-module then

$\|x\|_A = \|\langle x,x\rangle_A\|^{1/2}$ is a norm on X.

Proof :- We only need to show triangle inear.
For that note

$$\| \langle x+y, x+y \rangle_A \|_A = \| \langle x,x \rangle_A + \langle x,y \rangle_A + \langle y,x \rangle_A + \langle y,y \rangle_A \|$$

$$\leq \|x\|_A^2 + \|y\|_A^2 + 2\|x\|_A \|y\|_A$$

$$= (\|x\|_A + \|y\|_A)^2 .$$

[by CS $\|\langle x,y\rangle_A\|^2$
$= \|\langle x,y\rangle_A^* \langle x,y\rangle\|$
$\leq \|\langle x,x\rangle\| . \|\langle y,y\rangle\|$
$= \|x\|^2 . \|y\|^2$].

Defn :- Let A be a C^*- alg then a Hilbert right A-module is an innerproduct right A-module complete w.r.t the norm $\|x\|_A = \|\langle x,x\rangle_A\|^{1/2}$.

Cor :- (TO CS inear) Let X be an innerproduct right A-module then

$$\|x.a\|_A \leq \|x\|_A \|a\| .$$

Pf :-

$$\|x.a\|_A^2 = \cancel{\|\langle x, a\rangle} \|\langle x.a, x.a\rangle\|$$

$$= \|a^* \langle x,x\rangle a\|$$

$$\leq \|\langle x,x\rangle\| . \|a^* a\| = \|x\|_A^2 . \|a\|^2 .$$

Cor :- (To the above cor) Let X be an inner-product right A-module then completion of X is a Hilbert-A-module.

Propn :- The normed module X is nondegenerate i.e., the elements $x.a$ span a dense subspace of X. Indeed

$$X.\langle X, X\rangle_A = sp\{x.\langle y, z\rangle_A \mid x, y, z \in X\} \text{ is } \|\cdot\|_A \text{ dense}$$

in A.

Proof :- Let $\{u_\lambda\}$ be an approximate identity for the ideal $B = \overline{sp\{\langle x, y\rangle_A \mid x, y \in X\}}$

$$\|x - x.u_\lambda\|_A^2 = \|\langle x,x\rangle_A - \langle x,x\rangle_A u_\lambda - u_\lambda\langle x,x\rangle_A + u_\lambda\langle x,x\rangle_A u_\lambda\|$$

Given any $\varepsilon > 0$ $\exists$ u_{λ_0} s.t $\|x - x.u_{\lambda_0}\| < \varepsilon/2$.

[Because $\{u_\lambda\}$ is approximate identity means $\forall b \in B$]

$$\|b - b.u_\lambda\| \longrightarrow 0$$

$\exists$ $x_i, y_i \in X$ for $i = 1, \ldots n$ s.t

$$\left\|\sum_1^n \langle x_i, y_i\rangle - u_{\lambda_0}\right\|_A < \varepsilon/2\|x\|_A$$

$$\therefore \quad \left\|x - \sum x\langle x_i, y_i\rangle\right\|_A < \varepsilon.$$

Example :- Let H be a Hilbert space and $K(H)$ the C^*-alg of compact operators on H. Let $(|h\rangle\langle k|)$ be the operator given by

$$(|h\rangle\langle k|)(l) = \langle k, l\rangle h$$

Then with $T.h = T(h)$, H becomes a left $K(H)$ module.

H becomes a right $K(H)$ module provided we define $h.T = T^*.h$

with $\langle h, k \rangle_{K(H)} = |h\rangle\langle k|$

H becomes a left Hilbert $K(H)$ module.

Example:- Let T be a locally compact Hausdorff space and H a Hilbert space.

$$X = C_0(T, H) = \left\{ f : T \longrightarrow H \;\middle|\; \begin{array}{l} f \text{ is cont and} \\ t \longmapsto \|f(t)\| \in C_0(X) \end{array} \right\}$$

Then X is a Hilbert $C_0(T)$ module with

$$(f.a)(t) = a(t).f(t).$$

$$\langle f, g \rangle(t) = \langle f(t), g(t) \rangle$$

Example (Direct Sum) Suppose X and Y are Hilbert A-module is a right A-module. Then $Z = X \oplus Y$ is a right A-modules. Then $Z = X \oplus Y$ is a right A-module in the obvious way. We can define an A-valued inner product on Z by

$$\langle (x, y), (x', y') \rangle_A = \langle x, x' \rangle_A + \langle y, y' \rangle_A$$

Z is complete :-

$$\|x\|_A^2 = \|\langle x, x \rangle_A\| \le \|\langle x, x \rangle_A + \langle y, y \rangle_A\|$$

$$= \|(x, y)\|_A^2 \le \|x\|_A^2 + \|y\|_A^2.$$

In particular

$$\max(\|x\|_A, \|y\|_A) \le \|(x, y)\|_A \le \sqrt{\|x\|_A^2 + \|y\|_A^2}$$

Propn:- Let $\mathcal{A}$ be a C^*-alg. Then

$$H_{\mathcal{A}} = \left\{ \underline{a} = (a_i) : \sum^{\infty} a_i^* a_i \text{ converges in } \mathcal{A} \right\}$$

is a Hilbert-$\mathcal{A}$-module! with

$$\underline{a} \cdot x = (a_i \cdot x)$$

$$\langle \underline{a}, \underline{b} \rangle = \sum_{1}^{\infty} a_i^* b_i$$

Proof:- The formulas make sense :-

$$\sum_{i=m}^{n} (a_i \cdot x)^* (a_i \cdot x) = x^* \left(\sum_{i=m}^{n} a_i^* a_i \right) \cdot x$$

$$\leq \left\| \sum_{i=m}^{n} a_i^* a_i \right\| x^* x.$$

$\therefore \sum_{i=1}^{\infty} (a_i x)^* (a_i x)$ is convergent because

$$\left\| \sum_{i=m}^{n} a_i^* a_i \right\| < \varepsilon \text{ if } m, n \geq N.$$

$$\left\| \sum_{i=m}^{n} a_i^* b_i \right\|^2 \leq \left\| \sum_{i=m}^{n} a_i^* a_i \right\| \left\| \sum_{i=m}^{n} b_i^* b_i \right\|$$

Shows the series defining $\langle \underline{a}, \underline{b} \rangle$ converges.

Next we need to show completeness :-

Suppose $\{\underline{a}^{(n)}\} = \{(a_i^{(n)})\}$ is a Cauchy seq. in $H_{\mathcal{A}}$.

$\because$ ~~##~~ $\|a_i^{(n)}\|_{\mathcal{A}} \leq \|\underline{a}^{(n)}\|_{\mathcal{A}}$, each $\{a_i^{(n)}\}_n$ is a Cauchy seq in $\mathcal{A}$ converging to some a_i say.

We aim to show that $\underline{a} \in H_{\mathcal{A}}$ and $\underline{a}^{(n)} \longrightarrow \underline{a}$

To see that $\underline{a} \in H_{\mathcal{A}}$, we will show that $\forall \varepsilon > 0$ $\exists P$ s.t $m, n \geq P \Rightarrow \|\sum_{i=n}^{m} a_i^* a_i\| \leq \varepsilon^2$.

For $\underline{x} \in \prod_{i=1}^{\infty} \mathcal{A}$, $\|x\|_{n,m} = \|\sum_{i=n}^{m} x_i^* x_i\|^{1/2}$.

Note $\|x\|_{n,m} \leq \|x\|_{\mathcal{A}}$

As $\{\underline{a}^{(N)}\}$ is Cauchy,
$\exists N$ s.t $k, l \geq N \Rightarrow \|\underline{a}^{(k)} - \underline{a}^{(l)}\|_{\mathcal{A}} < \varepsilon/3$
Choose P s.t $P \geq N$, $\|\sum_{i=P}^{\infty} (a_i^{(N)})^* a_i^{(N)}\|^{1/2} < \varepsilon/3$.

Fix $m, n \geq P$.
$\exists M \geq N$ s.t $\|a - a^{(M)}\|_{n,m} < \varepsilon/3$.

Then

$$\|a_{n,m}\| \leq \|a - a^{(M)}\|_{n,m} + \|a^{(M)} - a^{(N)}\|_{n,m} + \|a^{(N)}\|_{n,m}$$

$$\leq \varepsilon/3 + \|\underline{a}^{(M)} - \underline{a}^{(N)}\|_{\mathcal{A}} + \|\sum_{i=P}^{\infty} (a_i^{(N)})^* a_i^{(N)}\|^{1/2}$$

$$\leq \varepsilon.$$

Since P depends only on ε. this shows $\underline{a} \in H_{\mathcal{A}}$.

Now we want to show that $\{\underline{a}^{(n)}\}$ converges to $\underline{a}$.

If $\varepsilon > 0$ $\exists$ N s.t $n, m \geq N \Rightarrow \| a^{(n)} - a^{(m)} \|_{A} \leq \varepsilon$

Then for any k.

$$\left\| \sum_{i=1}^{k} (a_i^{(n)} - a_i^{(m)})^* (a_i^{(n)} - a_i^{(m)}) \right\| \leq \varepsilon^2$$

Letting $m \to \infty$ gives

$$\left\| \sum_{i=1}^{k} (a_i^{(n)} - a_i)^* (a_i^{(n)} - a_i) \right\| \leq \varepsilon^2.$$

Since $\underline{a} \in H_A$, $\underline{a^{(n)}} - \underline{a} \in H_A$. and we get

$$\| a^{(n)} - a \| < \varepsilon \quad \forall n \geq N.$$

Maps on Hilbert modules :-

Defn :- Suppose X and Y are Hilbert A-modules. A fn $T : X \longrightarrow Y$ is called adjointable if $\exists\, T^* : Y \longrightarrow X$ s.t

$$\langle Tx, y\rangle_A = \langle x, T^*y\rangle_A \quad \forall x \in X, y \in Y$$

Lemma :- Every adjointable map $T : X \longrightarrow Y$ between A-modules is a bounded linear A-module map from X to Y.

Proof : C-S inequality shows that in any Hilbert A-module Z,

$$\|z\|_A = \sup \{\|\langle z, y\rangle_A\| : y \in Z,\ \|y\|_A \le 1\}$$

Hence $x = y$ in Z iff $\langle x, z\rangle_A = \langle y, z\rangle_A$, $\forall z \in Z$.

$$\langle T(x.a), y\rangle = \langle x.a, T^*(y)\rangle = a^*\langle x, T^*y\rangle$$

$$= \langle T(x).a, y\rangle \quad \forall y \in Y.$$

$$\therefore \quad T(x.a) = T(x).a$$

ie, T is A-linear.

T is bounded :-

Suppose $x_n \longrightarrow x$ in X & $Tx_n \longrightarrow z$ in Y.

Then $\forall y \in Y$

$$\langle Tx_n, y\rangle_A = \langle x_n, T^*y\rangle_A \longrightarrow \langle x, T^*y\rangle$$

$$= \langle Tx, y\rangle$$

On the other hand,

$$\langle Tx_n, y\rangle \longrightarrow \langle z, y\rangle.$$

$$\therefore \quad \langle Tx, y\rangle = \langle z, y\rangle \quad \forall y$$

$$\therefore \quad z = Tx$$

This shows graph of T is closed.

Hence T is bounded

Example:- Bounded linear maps need not be adjointable.

Let $A = C[0,1]$ and let $J = \{f \mid f(0)=0\}$

Then A & J are Hilbert A-modules.

Take $X = A \oplus J$.

Define $T : X \longrightarrow X$ by $T(f, g) = (g, 0)$

Then T is bounded with $\|T\| = 1$ and

T is A-linear.

If T had an adjoint and $T^*(1,0) = (f, g)$

then $\forall (h, k) \in X$.

$$\bar{k} = \langle T(h,k), (1,0)\rangle$$

$$= \langle (h,k), (f,g)\rangle$$

$$= \bar{h}.f + \bar{k}.g$$

$\therefore \quad f \equiv 0$ and $g \equiv 1$, which contradicts

$g(0) = 0$.

Thus T can not be adjointable.

Defn :- If X & Y are Hilbert A-modules, then $\mathcal{L}(X,Y)$ denote the space of adjointable maps from X to Y. $\mathcal{L}(X,X)$ is denoted by $\mathcal{L}(X)$.

Clearly $T \in \mathcal{L}(X,Y) \Longrightarrow T^* \in \mathcal{L}(Y,X)$
~~therefore~~ and $T^{**} = T$.
Thus $\mathcal{L}(X)$ is an involutive algebra

Propn :- If X is a Hilbert A-module, then $\mathcal{L}(X)$ is a C^*-alg wrt the operator norm.

Proof :- since $B(X)$ is a Banach algebra,

$$\|T^*T\| \le \|T^*\| . \|T\|$$

On the other hand from C-S inequ we get,

$$\|T^*T\| \ge \sup_{\|x\|\le 1} \|\langle T^*T(x), x\rangle\|$$

$$= \sup_{\|x\|\le 1} \|\langle Tx, Tx\rangle\|$$

$$= \|T\|^2.$$

$\therefore \quad \|T^*\| \ge \|T\|.$

$\therefore \|T\| = \|T^*\| \quad (\because T^{**} = T).$

$$\|T\|^2 \le \|T^*T\| \le \|T^*\| . \|T\| = \|T\|^2$$

implies $\|T^*T\| = \|T\|^2$.

The continuity of involution implies that $\mathcal{L}(X)$ is closed in $B(X)$ and hence a C^*-alg.

Cor:- If X is a Hilbert $\mathcal{A}$-module and $T \in \mathcal{L}(X)$ then $\langle Tx, Tx\rangle_{\mathcal{A}} \leq \|T\|^2 . \langle x, x\rangle_{\mathcal{A}}$ as elements of the C^*-alg $\mathcal{A}$.

Proof:- $T^*T \leq \|T\|^2 . I$.

$\therefore \exists\, S$ s.t $\|T\|^2 . I - T^*T = S^*S$.

$$\|T\|^2 . \langle x, x\rangle_{\mathcal{A}} - \langle Tx, Tx\rangle_{\mathcal{A}}$$

$$= \langle (\|T\|^2 . I - T^*T) . x, x\rangle_{\mathcal{A}}$$

$$= \langle S^*Sx, x\rangle_{\mathcal{A}} = \langle Sx, Sx\rangle_{\mathcal{A}} \geq 0.$$

Defn:- Given Hilbert $\mathcal{A}$-modules X & Y we define "rank-1" operators as follows:- Let $x \in X$, $y \in Y$ then $(|y\rangle\langle x|) : X \to Y$ is the map given by $(|y\rangle\langle x|)(z) = y\langle x, z\rangle_{\mathcal{A}}$.

Note:- $(|y\rangle\langle x|)^* = (|x\rangle\langle y|)$.

$K(X,Y)$ is the closed subspace of $\mathcal{L}(X,Y)$ spanned by $\{(|y\rangle\langle x|) : x \in X, y \in Y\}$.

$K(X,X)$ is denoted by $K(X)$ and its elements are called compact operators, even though its elements are not compact operators.

Propn :- $K(X)$ is an ideal in $\mathcal{L}(X)$.

Proof :- Let $T \in \mathcal{L}(X)$ then

$$T(|x\rangle\langle y|) = |Tx\rangle\langle y|$$

So, $K(X)$ is a left ideal.

$(|x\rangle\langle y|)^* = (|y\rangle\langle x|)$ implies $K(X)$ is $*$-closed

we conclude that $K(X)$ is an ideal.

Example :- Let $X = A$ ~~~~ considered as a right A-module.

Define $L : A \longrightarrow \mathcal{L}(A)$.

$$L_a(x) = a.x$$

$$\langle L_a(x), y\rangle = \langle ax, y\rangle = x^* a^* y = \langle x, a^*.y\rangle$$
$$= \langle x, L_{a^*} y\rangle$$

$\therefore$ L_a is adjointable with

$$(L_a)^* = L_{a^*}.$$

$$L_a L_b = L_{ab}.$$

$\therefore$ L is a $*$-homomorphism, hence $\|L_a\| \leq \|a\|$

$$\|L_a(a^*)\| = \|a\|.\|a^*\|$$

$\therefore$ $\|L_a\| \geq \|a\|$

Thus L is an isometry.

$$(|a\rangle\langle b|)(c) = ab^* c = L_{ab^*}(c).$$

Thus $K(A)$ is inside the closure of the image of L.

Since A always has an approximate identity $L_a \in K(A)$ $\forall a$

So, $L : A \longrightarrow K(A)$ is an isomorphism.

Defn:- A Hilbert $\mathcal{A}$-module X is called full if $\overline{sp\langle X, X\rangle_{\mathcal{A}}} = \mathcal{A}$.

Lemma:- Let $T : X \longrightarrow X$ be a linear map. Then T is a +ve element of $\mathcal{L}(X)$ iff $\langle T(x), x\rangle_{\mathcal{A}} \geq 0$ $\forall x \in X$.

Pf:- If $T \geq 0$ in $\mathcal{L}(X)$ then $T = S^{*}S$ and

$$\langle Tx, x\rangle_{\mathcal{A}} = \langle Sx, Sx\rangle_{\mathcal{A}} \geq 0.$$

Now suppose $\langle Tx, x\rangle_{\mathcal{A}} \geq 0$ $\forall x \in X$.

$$4\langle x, Ty\rangle_{\mathcal{A}} = \sum_{k=0}^{3} i^k \langle x + i^k y, T(x + i^k y)\rangle_{\mathcal{A}}$$

and $\langle Tz, z\rangle_{\mathcal{A}} = \langle z, Tz\rangle_{\mathcal{A}}$, $\forall z \in X$.

It follows that $\langle Tx, y\rangle_{\mathcal{A}} = \langle x, Ty\rangle_{\mathcal{A}}$.

Thus T is adjointable and $T^{*} = T$.

Now by functional calculus we can write $T = S - R$. with $S, R \geq 0$ in $\mathcal{L}(X)$. and $SR = RS = 0$.

Then for all $x \in X$,

$$0 \leq \langle TRx, Rx\rangle_{\mathcal{A}} = -\langle R^3 x, x\rangle_{\mathcal{A}}$$

Since $R^3 \geq 0$, it follows that $\langle R^3 x, x\rangle_{\mathcal{A}} = 0$ $\forall x$.

$R^3 = 0$ by polarization. identity ~~and~~ and $R = 0$.. Thus $T = S \geq 0$.

Ex:- $T \geq 0$ in $\mathcal{L}(X)$ implies
$$\|T\| = \sup\{\|\langle Tx, x\rangle_A\| : \|x\|_A \leq 1\}$$
(If $T = S^*S$, then R.H.S $= \|S\|^2 = \|S^*S\| = \|T\|$)

Lemma:- Let A be a C^*-algebra and suppose that X is a right Hilbert A-module. Then X is a full left Hilbert $K(X)$ module with respect to the natural left action $T.x = T(x)$ and the inner product
$${}_{K(X)}\langle x, y\rangle = |x\rangle\langle y|.$$
Moreover the norms
$$\|x\|_A = \|\langle x, x\rangle_A\|^{1/2} \text{ \& } \|x\|_{K(X)} = \|{}_{K(X)}\langle x, x\rangle\|^{1/2}$$
coincide.

Proof:- We need to verify left hand versions of properties (a) - (e). The fullness is clear from defn of $K(X)$.
$$T\,{}_{K(X)}\langle x, y\rangle = T(|x\rangle\langle y|) = |Tx\rangle\langle y| = {}_{K(X)}\langle Tx, y\rangle$$
$(|x\rangle\langle y|)^* = |y\rangle\langle x|$ gives $({}_{K(X)}\langle x, y\rangle)^* = {}_{K(X)}\langle y, x\rangle$

$$(*) \text{——} \langle {}_{K(X)}\langle x, x\rangle . y, y\rangle_A = \langle x\langle x, y\rangle_A, y\rangle_A$$
$$= (\langle x, y\rangle_A)^* \langle x, y\rangle_A \geq 0$$

implies ${}_{K(X)}\langle x, x\rangle \geq 0$.

Now suppose ${}_{K(X)}\langle x, x\rangle = 0$.

then (*) implies $\langle x, y\rangle_{\mathcal{A}} = 0 \ \forall y$

Taking $y = x$, we conclude $x = 0$.

To compute the norm coming from the left inner product we use CS inequ.

$$\langle {}_{K(X)}\langle x, x\rangle \cdot y, y\rangle_{\mathcal{A}} = (\langle x, y\rangle_{\mathcal{A}})^{*}(\langle x, y\rangle_{\mathcal{A}}).$$

$$\leq \|\langle x, x\rangle_{\mathcal{A}}\| \cdot \langle y, y\rangle_{\mathcal{A}}$$

Hence $\quad \|{}_{K(X)}\langle x, x\rangle\| \leq \|\langle x, x\rangle_{\mathcal{A}}\|$.

On the other hand $y = x$ gives

$$\|\langle {}_{K(X)}\langle x, x\rangle x, x\rangle_{\mathcal{A}}\| = \|\langle x, x\rangle_{\mathcal{A}}\|^{2}.$$

$$\therefore \quad \|{}_{K(X)}\langle x, x\rangle\| \geq \|\langle x, x\rangle_{\mathcal{A}}\|.$$

Multiplier algebras :-

Defn:- An ideal $\mathcal{I}$ in a C^*-algebra $\mathcal{A}$ is essential if $\mathcal{I}$ has nonzero intersection with every other nonzero ideal $\mathcal{A}$.

Lemma:- An ideal $\mathcal{I}$ is essential iff $a.\mathcal{I} = \{0\} \Rightarrow a = 0$.

Proof:- For $a \in \mathcal{A}$, let $J_a = \overline{\mathcal{A} a \mathcal{A}} = \overline{\text{span}\{bac : b, c \in \mathcal{A}\}}$ be the ideal generated by a.

Note for any two ideals J_1 & J_2 we have

$J_1 \cap J_2 = J_1 . J_2$

This is so because clearly $J_1 . J_2 \subseteq J_1 \cap J_2$.

For the other inclusion fix an approximate identity $\{u_\lambda\}$ for J_2.

Now given $x \in J_1 \cap J_2$, $x u_\lambda \in J_1 . J_2$

& $x = \lim x u_\lambda \in J_1 . J_2$

Thus $J_1 \cap J_2 \subseteq J_1 . J_2$.

Claim:- $J_a . \mathcal{I} = \{0\}$ iff $a.\mathcal{I} = \{0\}$.

Pf:- Only if:- $a.\mathcal{I} \subseteq J_a . \mathcal{I} = \{0\}$

If part:- If $a.\mathcal{I} = \{0\}$ then $a.b.x = 0 \ \forall b \in \mathcal{A}, x \in \mathcal{I}$.

$\therefore J_a . \mathcal{I} = \{0\}$.

It follows that $J_a \cap \mathcal{I} = \{0\}$ iff $a.\mathcal{I} = \{0\}$.

Thus if $\mathcal{I}$ is essential and $a.\mathcal{I} = 0$ then $J_a = \{0\}$ and $a = 0$.

Conversely suppose $a.\mathcal{J} = \{0\}$ implies $a = 0$.
If J is a nonzero ideal and $a \in J \setminus \{0\}$.

$$a \neq 0 \Rightarrow a.\mathcal{J} \neq \{0\} \Rightarrow J_a \cap \mathcal{J} \neq \{0\} \Rightarrow J \cap \mathcal{J} \neq \{0\}.$$

<u>Defn</u>:- A unitization of a C^*-alg A is a C^*-alg B with identity and an injective homo $i : A \longrightarrow B$ such that $i(A)$ is an essential ideal of B.

<u>Remark</u>:- If A is unital then only unitization is A-itself. For if ~~B~~ A is an ideal in B and $b \in B \setminus A$ then $b.1 \in A$, $b - b.1 \neq 0$. and $(b - b.1).A = 0$ so A is not essential.

<u>Example</u>:- Let A be a C^*-alg without identity.
$A^+ = A \oplus \mathbb{C}$ is a $*$-alg. with

$$(a+\lambda)(b+\mu) = ab + \lambda b + \mu a + \lambda\mu$$

$$\text{and} \quad (a+\lambda)^* = a^* + \bar{\lambda}$$

To give A^+ a C^*-norm
consider the homomorphism

$$L : A^+ \longrightarrow B(A) \text{ given by}$$

$$L_{(a,\lambda)}(b) = ab + \lambda b$$

and define

$$\|(a,\lambda)\| = \|L_{(a,\lambda)}\|_{op}$$

L is one to one :–

Suppose $ab + \lambda b = 0 \quad \forall b \in \mathcal{A}$

If $\lambda \neq 0$, $\left(-\frac{a}{\lambda}\right) . b = b \quad \forall b \in \mathcal{A}$. Ex.

and hence $\left(-\frac{a}{\lambda}\right)$ is a unit for $\mathcal{A}$ (Why?)

This contradicts the hypothesis on $\mathcal{A}$.

If $\lambda = 0$, then $ab = 0 \quad \forall b \in \mathcal{A}$

In particular $aa^* = 0$. But then $\|a\|^2 = 0$.

The inclusion $a \longmapsto (a, 0) \quad \mathcal{A} \hookrightarrow \mathcal{A}^+$ is

isometric :–

Since $\|ab\| \leq \|a\|.\|b\|$

$\|L_{(a,0)}\|_{op} \leq \|a\|$

The eqn $\|aa^*\| = \|a\|^2$ implies $\|a\| \leq \|L_{(a,0)}\|_{op}$

It only remains to check that the norm on $\mathcal{A}^+$ satisfies the C^*-identity.

i.e, $\|(a, \lambda)^*(a, \lambda)\| = \|(a, \lambda)\|^2$.

For this let $\varepsilon > 0$.

By defn of operator norm $\exists b \in \mathcal{A}$ s.t

$\|b\| = \|(b, 0)\| = 1$

$\|ab + \lambda b\| \geq \|(a, \lambda)\|.(1 - \varepsilon)$.

$$(1-\varepsilon)^2 \cdot \|(a,\lambda)\|^2 \leq \|(ab+\lambda b)\|^2$$
$$= \|(ab+\lambda b)^*(ab+\lambda b)\|$$
$$= \|(b^*,0)(a^*,\bar{\lambda}).(a,\lambda)(b,0)\|$$
$$\leq \|(b^*,0)\|.\|(a^*,\bar{\lambda})(a,\lambda)\|.\|(b,0)\|$$
$$= \|(a^*,\bar{\lambda}).(a,\lambda)\|$$

Since ε is arbitrary we get

$$(**) \qquad \|(a,\lambda)\|^2 \leq \|(a,\lambda)^*(a,\lambda)\| \leq \|(a,\lambda)^*\|.\|(a,\lambda)\|$$

$$\therefore \quad \|(a,\lambda)\| \leq \|(a,\lambda)^*\|$$

$$\therefore \quad \|(a,\lambda)^*\| = \|(a,\lambda)\|$$

So, we get C^* identity from $(**)$.

<u>Example</u>:- $A = C_0(X)$. A compact Hausdorff sp. Y is called a compactification if $\exists\, i : X \hookrightarrow Y$ with $i(X)$ a dense open subset.

Then $\quad i_* : C_0(X) \longrightarrow C(Y)$

$$(i_* f)(y) = \begin{cases} 0 & \text{if } y \notin i(X) \\ f(x) & \text{if } y = i(x) \text{ for some } x \end{cases}$$

Defn :- A unitization $i : A \longrightarrow B$ is called maximal if for every embedding $j : A \longrightarrow C$ with $j(A)$ an essential ideal of C $\exists\ \varphi : C \longrightarrow B$ s.t

$$\begin{tikzcd} & B \\ A \arrow[ur, "i"] \arrow[dr, "j"'] & \\ & C \arrow[uu, "\varphi"'] \end{tikzcd}$$

Propn :- The map $L : \mathcal{A} \longrightarrow \mathcal{L}(\mathcal{A})$ is a unitization

Proof :- We have already seen $L : \mathcal{A} \longrightarrow \mathcal{K}(\mathcal{A})$ is an isomorphism and $\mathcal{K}(\mathcal{A})$ is an ideal. Only thing we need to show is $K(\mathcal{A})$ is essential. But if $T \in \mathcal{L}(\mathcal{A})$ satisfies then

$$TK = 0 \quad \forall K \in K(\mathcal{A}),$$

$$0 = T(|b\rangle\langle c|) = |Tb\rangle\langle c| \quad \forall b, c \in \mathcal{A}.$$

This implies $Tb = 0$ $\forall b$ and $T = 0$.

Theorem :- For any C^*-algebra $\mathcal{A}$ the unitization $L : \mathcal{A} \longrightarrow \mathcal{L}(\mathcal{A})$ is maximal. It is unique: if $j : \mathcal{A} \longrightarrow B$ is another maximal unitization then there is an isomorphism φ of B onto $\mathcal{L}(\mathcal{A})$ s.t $\varphi \circ j = L$

Defn :- We refer to $\mathcal{L}(\mathcal{A})$ as the multiplier algebra of $\mathcal{A}$.

Proof of theorem needs some preparation.

<u>Defn</u>:- Suppose that B is a C^*-algebra and X is a Hilbert A-module. A homomorphism $\alpha : B \longrightarrow \mathcal{L}(X)$ is nondegenerate if

$$\alpha(B).X = \text{span}\{\alpha(b).x \mid b \in B, x \in X\}$$

is dense in X.

<u>Proposition</u>:- Let A, B, C be C^*-algebras, X a Hilbert A-module, $i : B \longrightarrow C$ an injective homomorphism onto an ideal in C. If $\alpha : B \longrightarrow \mathcal{L}(X)$ is a nondegenerate homomorphism. Then there is a unique homomorphism $\bar{\alpha} : C \longrightarrow \mathcal{L}(X)$ such that $\bar{\alpha} \circ i = \alpha$.
If $i(B)$ is an essential ideal and α is injective, then $\bar{\alpha}$ is injective.

<u>Pf</u>:- Wlg we can assume that B is an ideal in C. Let $\{e_\lambda\}$ be an approximate identity for B. Then if $c \in C$, $b_1, \ldots, b_n \in B$ and $x_1, \ldots, x_n \in X$

$$\begin{aligned}
\left\| \sum_{i=1}^{n} \alpha(cb_i)(x_i) \right\| &= \lim_{\lambda} \left\| \sum_{i=1}^{n} \alpha(ce_\lambda b_i)(x_i) \right\| \\
&= \lim_{\lambda} \left\| \alpha(ce_\lambda) \sum_{i=1}^{n} \alpha(b_i) x_i \right\| \\
&\leq \|c\| . \left\| \sum_{i=1}^{n} \alpha(b_i) x_i \right\|
\end{aligned}$$

Thus $\left(\sum_{1}^{n} \alpha(b_i) x_i\right) \longmapsto \left(\sum_{i=1}^{n} \alpha(cb_i) x_i\right)$

is well defined and bounded on $\alpha(B) \cdot X$ and so extends to a bounded operator $\bar{\alpha}(c)$ on X, which is in $\mathcal{L}(X)$ because $\bar{\alpha}(c^*)$ is an adjoint.

Clearly $\bar{\alpha}$ is a homomorphism and it is unique because elements of the form $\sum_{1}^{n} \alpha(b_i) x_i$ are dense in X.

Finally if α is injective and B is essential, then

$$\ker(\bar{\alpha}) \cap B = \ker(\alpha) \cap B = \{0\} \text{ implies } \ker \bar{\alpha} = \{0\}.$$

Cor:- If $\varphi : B \longrightarrow M(A)$ is a nondegenerate homomorphism, then $\exists!$ homomorphism $\bar{\varphi} : M(B) \longrightarrow M(A)$ such that $\bar{\varphi}(b) = \varphi(b)$ $\forall b \in B$.

Proof:- Take $X = A$, $C = M(B)$ and $i : B \longrightarrow M(B)$ the inclusion of B in $M(B)$.

Lemma:- If $i : A \hookrightarrow B$ is a maximal unitization and if $j : A \hookrightarrow C$ embeds A as an essential ideal, then there is only one homomorphism $\varphi : C \longrightarrow B$ s.t $\varphi \circ j = i$. and it is injective.

Proof :– φ is injective :–

$\ker(\varphi) \cap j(A) = \{0\}$ and $j(A)$ is essential

Therefore, $\ker(\varphi) = \{0\}$.

Suppose $\psi : C \longrightarrow B$ is another such homomorphism

If $c \in C$,

$(\psi(c) - \varphi(c))\, i(a) = \varphi(c j(a)) - \psi(c j(a))$ $\quad \forall a \in A$.

$= 0$, $\because$ $c j(a) \in j(A)$.

and $\varphi = \psi$ on $j(A)$.

Thus $(\psi(c) - \varphi(c)) . i(A) = \{0\}$.

Since $i(A)$ is an essential ideal in B we deduce that $\varphi(c) = \psi(c)$ $\forall c \in C$.

Proof of theorem :–

Uniqueness :– The maximality of $\mathcal{L}(A)$ gives $\varphi : B \longrightarrow \mathcal{L}(A)$ and maximality of B gives $\psi : \mathcal{L}(A) \longrightarrow B$. Then by uniqueness They are inverse to each other.

Maximality of $\mathcal{L}(A)$:–

Suppose $j : A \longrightarrow C$ embeds as an essential ideal. We know $L : A \longrightarrow \mathcal{L}(A)$ is nondegenerate. By the previous propn j extends to a monomorphism $I : C \longrightarrow \mathcal{L}(A)$ s.t $I \circ j = L$.

Proposition :- Let A, C be C^*-algebras and X - a Hilbert C-module. Let $\alpha : A \longrightarrow \mathcal{L}(X)$ be an injective nondegenerate homomorphism. Then α extends to an isomorphism of $M(A)$ onto

$$B = \{ T \in \mathcal{L}(X) : T.\alpha(A) \subseteq \alpha(A), \alpha(A).T \subseteq \alpha(A) \}.$$

Proof :- $\alpha(A)$ is an ideal in B and is essential because if $T\alpha(A) = \{0\}$, then $T\alpha(A).X = \{0\}$.

This forces $T(X) = \{0\}$. by nondegeneracy & $T = 0$.

So, we need to show if $j : A \longrightarrow D$ is an embedding as an essential ideal then $\exists\ \varphi : D \longrightarrow B$ extending j.

We know $\exists\ \bar{\alpha} : D \longrightarrow \mathcal{L}(X)$ s.t $\bar{\alpha} \circ j = \alpha$.

Suffices to show that $\bar{\alpha}(D) \subseteq B$.

But if $d \in D$, and $a \in A$, then

$$\bar{\alpha}(d)\alpha(a) = \bar{\alpha}(d)\,\bar{\alpha}(j(a)) = \bar{\alpha}(dj(a)) \in \alpha(A)$$

because $dj(a) \in j(A)$.

Cor :- If X is a Hilbert A-module, then $i : K(X) \longrightarrow \mathcal{L}(X)$ is a maximal unitization of $K(X)$. Therefore $\mathcal{L}(X) \cong M(K(X))$.

Pf :- we need to show i is nondegenerate.

Let $\{u_\lambda\}$ be an approximate identity for $\langle X, X\rangle_A$.

Claim :- $\forall x \in X, \ \lim_{\lambda} x \cdot u_\lambda = x$.

Proof :- $\| \langle x \cdot u_\lambda - x, x u_\lambda - x \rangle \|$

$$\leq \| u_\lambda \langle x, x\rangle u_\lambda + \langle x, x\rangle - u_\lambda \langle x, x\rangle - \langle x, x\rangle u_\lambda \|$$

$$\leq \| u_\lambda \langle x, x\rangle - \langle x, x\rangle \| \cdot \| u_\lambda \| + \| \langle x, x\rangle - u_\lambda \langle x, x\rangle \| \longrightarrow 0.$$

$$\therefore \quad \overline{K(X) \cdot X} \supseteq \overline{X \langle X, X\rangle} \supseteq X.$$