Perron-Frobenius theorem

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1. topology (Brouwer fixed-point theorem)
2. Graph theory
3. probability theory (finite-state Markov chains)
4. von Neumann algebras (subfactors)

We first state a simpler[1] special case of the theorem, due to **Perron**. In the sequel, we shall find it convenient to use the non-standard notation $B > 0$ (resp., $B \geq 0$) for any (possibly even rectangular) matrix with positive (resp., non-negative) entries.

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Theorem

Let $A = ((a^i_j)) > 0$ *be a square matrix, and let* $\lambda^*(A) = r(A) = \sup\{|\lambda| : \lambda$ *an eigenvalue of* $A\}$. *Then*

1. $\lambda^*(A)$ *is an eigenvalue of* A *of (algebraic, hence also geometric) multiplicity one, and (a suitably scaled version* v^* *of) the corresponding eigenvector has strictly positive entries;*
2. $|\lambda| < \lambda^*(A)$ *for all eigenvalues of* A.

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Proof: Let $\Delta = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_j \geq 0 \forall j$ and $\sum_{j=1}^n x_j = 1\}$ be the standard simplex in $\mathbb{R}^n$.

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One proof rests on the observation that

$$\begin{aligned}\lambda^*(A) &= \inf\{\lambda > 0 : \exists 0 \neq v \geq 0 \text{ such that } Av \leq \lambda v\} \\ &= \inf\{\lambda > 0 : \exists v \in \Delta \text{ such that } Av \leq \lambda v\} \ ,\end{aligned}$$

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The Frobenius extension of the theorem relaxes the strict positivity assumption on A. Call A *irreducibile* if it satisfies either of the following equivalent conditions:

1. there does not exist a permutation matrix P such that PAP' has the form $A_1 \oplus A_2$ for some matrices A_i of strictly smaller size;
2. $\forall\ 1 \leq i, j \leq n$, there exists some $m > 0$ such that $(A^m)^i_j > 0$.

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A good example to bear in mind is the *cyclic permutation matrix* U with

$$u^i_j = \begin{cases} 1 & \text{if } i = j + 1 (mod\ n) \\ 0 & \text{otherwise} \end{cases}$$

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1. $\lambda^*(A)$ *is an eigenvalue of A of (algebraic, hence also geometric) multiplicity one, and (a suitably scaled version v^* of) the corresponding eigenvector has strictly positive entries;*
2. *The only non-negative eigenvectors of A are multiples of v^*;*
3. *If*

$$|\{\lambda \in \mathbb{C} : \lambda \text{is an eigenvalue of } A \text{ such that } |\lambda| = \lambda^*(A)\}| = k ,$$

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The following re-formulation of irreducibility is instructive and useful (especially in applications to graphs and their adjacency matrices): A is irreducible precisely if for all $i, j \leq n$, there exists some $m \geq 0$ such that $(A^m)^i_j > 0$.

In lieu of a proof of the PF-theorem, we shall deduce the the existence of the Perron-Frobenius eigenvector from the Brouwer fixed point theorem. This latter fundamental result from topology asserts that any continuous self-map of the unit ball $\mathbb{B}^n$ (or equivalently, any compact convex set in $\mathbb{R}^n$) admits a fixed point.

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If $A \geq 0$ is irreducible, and if $v \in \Delta$, then $Av \neq 0$. (*Reason:* If $v_j \neq 0$, and if $(A^m)^i_j > 0$, then it is clear that $A^m v \neq 0$.) Hence $\|Av\|_1 = \sum_{j=1}^n (Av)_j > 0$. Define $f : \Delta \to \Delta$ by $f(v) = (\|Av\|_1)^{-1} Av$. Let v^* denote the fixed point guaranteed by Brouwer.

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Actually, there is an interesting proof of the Brouwer fixed point theorem using the PF theorem, which may be found in an article by the economist Scarf.

A graph $G = (V, E)$ consists of a finite set V of edges and a finite set E of edges connecting pairs of vertices. If $V = \{v_1, \cdots, v_n\}$, the so-called *adjacency matrix* $A(G)$ is obtained by decreeing that a^i_j is the number of edges joining v_i and v_j. By definition, it is seen that $A(G)$ is an $n \times n$ symmetric matrix with non-negative integer entries.

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A little thought reveals that the (i, j)-th entry of A^k counts the number of paths of length k between v_i and v_j. Hence, it is a consequence of our earlier reformulation of irreducibility of an $A \geq 0$, that $A(G)$ is irreducible precisely when the graph G is connected.

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Since the eigenvalues of a symmetric matrix are real, we find that the Perron-Frobenius eigenvalue of $A(G)$, for a connected graph G, is the largest eigenvalue of $A(G)$ as well as its largest singular value; this is an important isomorphism invariant of G. For instance:

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the PF eigenvalue of $A(G)$ less than 2 if and only if

$$G \in \{A_m, D_n, E_k : m \geq 2, n \geq 3, k = 6, 7, 8\} .$$

An *n-state Markov chain* is described by an $n \times n$ matrix $P = ((p^i_j))$. Here we are modeling a particle which can be in any one of n possible states at any given day, with the probability that the particle making a transition from site i to site j on any given day being given by p^i_j; thus, we write

$$p^i_j = Prob(X_{N+1} = j | X_N = i) \ \forall 1 \leq i, j \leq n, N \geq 0 \ ,$$

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If P is to have an interpretation as a *transition probability matrix*, it must clearly satisfy $\sum_{j=1}^n p^i_j = 1 \ \forall i$, or equivalently, $Pv = v$ where v is the vector with all coordinates equal to 1. In particular, v is **the** PF-eigenvector of P. Since P and P' have the same eigenvalues, we see that also $\lambda^*(P') = 1$.

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Let π denote the PF eigenvector of P', so normalised that $\sum_{i=1}^n \pi_i = 1$. This π has the interpretation of a *stationery distribution* for the process $\{X_N : N \geq 0\}$, meaning that $Prob(X_N = i) = \pi_i \ \forall i \leq n, N \geq 0$.

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It must be noticed that the above use of the Perron-Frobenius theorem is not really vald if P is not irreducible. Not surprisingly, the best behaved Markov chains are the ones with irreducible transition probability matrices.

Definition

A **fusion algebra** is a (usually finite-dimensional, for us) complex, associative, involutive algebra $\mathbb{C}\mathcal{G}$ equipped with a distinguished basis $\mathcal{G} = \{\alpha_i : 0 \leq i < n\}$ which satisfies:

- α_0 is the multiplicative identity of $\mathbb{C}\mathcal{G}$.
- The 'structure constants' given by $\alpha_i \alpha_j = \sum_{k=0}^{n-1} N_{ij}^k \alpha_k$ are required to be non-negative integers.
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It is a consequence of the axioms that we also have

$$N_{ij}^k = N_{k\bar{j}}^i \tag{2}$$

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4. The *raison d'etre* for our interest in this notion lies in another family of typically infinite-dimensional fusion algebras, but often with many interesting finite-dimensional 'sub-fusion algebras', which arises in the theory of II_1 factors; we shall now briefly pause for a digression into these beautiful objects.

A *von Neumann algebra* is a Banach $*$-algebra (in fact a C^*-algebra), which happens to be the Banach dual space of a canonically determined separable Banach space. Much of the rich structure of von Neumann algebras stems from this canonically inherited (so-called σ-weak) topology in which its norm-unit ball is compact, thanks to Alaoglu.

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The collection $\mathcal{L}(\mathcal{H})$ of all bounded operators on a separable Hilbert space is a von Neumann algebra. A $*$-homomorphism between von Neumann algebras is said to be *normal* if it is continuous when domain and range are equipped with the σ-weak topologies. By a module over a von Neumann algebra is meant a separable Hilbert space $\mathcal{H}$ equipped with a normal homomorphism from M into $\mathcal{L}(\mathcal{H})$. The *Gelfand-Naimark theorem* ensures that every von Neumann algebra admits a 'faithful' module.

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- If ${}_M\mathcal{H}_N$ and ${}_N\mathcal{H}_P$ are bifinite bimodules, there is a canonically associated bifinite $M - P$ bimodule $\mathcal{H} \otimes_N \mathcal{K}$ such that

$$\begin{aligned} dim_{M-}(\mathcal{H} \otimes_N \mathcal{K}) &= dim_{M-}(\mathcal{H}) dim_{N-}(\mathcal{K}) \\ dim_{-P}(\mathcal{H} \otimes_N \mathcal{K}) &= dim_{-N}(\mathcal{H}) dim_{-P}(\mathcal{K}) \end{aligned}$$

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- If $\mathcal{G}_N$ denotes the collection of isomorphism classes of bifinite $N - N$ bimodules, then $\mathbb{C}\mathcal{G}_N$ is a typically infinite-dimensional fusion algebra.

dimension function

Theorem

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Proof: Define an inner-product on $\mathbb{C}\mathcal{G}$ by demanding that $\mathcal{G}$ is an orthonormal basis. For each $\alpha \in \mathcal{G}$, let λ_α be the operator, on $\mathbb{C}\mathcal{G}$, of left multiplication by α. With respect to the ordered basis $\mathcal{G} = \{\alpha_0, \cdots, \alpha_{n-1}\}$, we may identify λ_{α_k} with the matrix $L(k)$ given by $L(k)^i_j = \langle \alpha_k \alpha_j, \alpha_i \rangle = N^i_{kj}$. (Notice that equation (1) says that $L(i)^* = L(\bar{i})$.)

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Proof: Define an inner-product on $\mathbb{C}\mathcal{G}$ by demanding that $\mathcal{G}$ is an orthonormal basis. For each $\alpha \in \mathcal{G}$, let λ_α be the operator, on $\mathbb{C}\mathcal{G}$, of left multiplication by α. With respect to the ordered basis $\mathcal{G} = \{\alpha_0, \cdots, \alpha_{n-1}\}$, we may identify λ_{α_k} with the matrix $L(k)$ given by $L(k)^i_j = \langle \alpha_k \alpha_j, \alpha_i \rangle = N^i_{kj}$. (Notice that equation (1) says that $L(i)^* = L(\bar{i})$.)

Similarly, the operator ρ_{α_k}, on $\mathbb{C}\mathcal{G}$, of right multiplication by α_k, will be represented by a matrix $R(k)$ of non-negative integral entries; in fact,

$$R(k)^i_j = \langle \alpha_j \alpha_k, \alpha_i \rangle = N^i_{jk} \ .$$

(Again equation (2) says that $R(j)^* = R(\bar{j})$.)

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$$R(k)^i_j = \langle \alpha_j \alpha_k, \alpha_i \rangle = N^i_{jk} \ .$$

(Again equation (2) says that $R(j)^* = R(\bar{j})$.)

Assertion: If $R = \sum_{k=0}^{n-1} R(k)$, then $R > 0$ meaning of couse that $R^i_j > 0 \ \forall i, j$.

If possible, suppose $R^i_j = 0$ for some i, j. Since $R^i_j = \sum_k R(k)^i_j = \sum_k N^i_{jk}$, the assumed non-negativity of the structure constants then implies that $N^i_{jk} = 0 \; \forall k$. Hence

$$\begin{aligned} \alpha_{\bar{j}}\alpha_i &= \sum_k N^k_{\bar{j}i}\alpha_k \\ &= \sum_k N^i_{jk}\alpha_k \quad (\text{ by equation}(1)) \\ &= 0\ ; \end{aligned}$$

and so,

$$\begin{aligned} 0 &= \alpha_j(\alpha_{\bar{j}}\alpha_i)\alpha_{\bar{i}} \\ &= (\alpha_j\alpha_{\bar{j}})(\alpha_i\alpha_{\bar{i}})\ . \end{aligned}$$

On the other hand, the coefficient of α_0 in $(\alpha_j\alpha_{\bar{j}})(\alpha_i\alpha_{\bar{i}})$ is seen to be $\sum_k N^k_{j\bar{j}} N^{\bar{k}}_{i\bar{i}}$ which is at least 1. This contradiction establishes the Assertion.

Finally, since $L(k)$ clearly commutes with any $R(j)$, we find that $L(k)$ commutes with R and must consequently leave the eigenspace of R corresponding to its Perron eigenvalue. This latter space is spanned by the PF eigenvector, say v, of R; hence $L(k)v = d_k v$ for some d_k.

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Then,

$$\begin{aligned} d_i d_j v &= d_j L(i) v \\ &= L(i)(d_j v) \\ &= L(i)L(j)v \\ &= (\sum_k N_{ij}^k L(k))v \\ &= \sum_k N_{ij}^k d_k v \end{aligned}$$

and it is easy to see that the linear extension of the function $\mathcal{G} \ni \alpha_i \mapsto d_i \in (0, \infty)$ is the desired dimension function.

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The proof of uniqueness of the dimension function follows by observing that the vector with i-th coordinate d_i - for any potential dimension function - is a positive eigenvector of R, whose 0-th coordinate is 1. So uniqueness is also a conswequence of the PF theorem. □

The simplest example of a fusion algebra where the dimension function assumes non-integral values is given by $\mathcal{G} = \{1, \alpha\}$, where $\alpha = \alpha^*$ and $\alpha^2 = 1 + \alpha$. The dimension function must satisfy $d(\alpha) = \phi$, where $\phi^2 = 1 + \phi$ so that $\phi = \frac{1+\sqrt{5}}{2}$ is the golden mean! This fusion algebra is the first of a whole family of fusion algebras arising from the theory of subfactors, which give meaning to certain irreducible bimodules having interesting dimension values like ϕ'!

References

1. R.B. Bapat and T.E.S. Raghavan, *Non-negative matrices and applications*, Cambridge Univ. Press, 1997.
2. F.R. Gantmacher, *The theory of matrices, Volume 2*, Amer. Math. Soc., 1959.
3. H. Scarf, *The computation of equilibrium prices: an exposition*, Cowles Foundation paper 41.
4. V.S. Sunder, II_1 *factors, their bimodules and hypergroups*, Trans. Amer. Math. Soc., vol. **330** no. 1, 1992.