Operator algebras - stage for non-commutativity
(Panorama Lectures Series)
V. The Jones polynomial invariant of knots

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Recap

Recall that if $N \subset M$ is a subfactor with finite index, say τ^{-1}, and basic construction tower

$$N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 \subset \cdots \subset M_{n-1} \subset M_n \subset^{e_{n+1}} M_{n+1} \subset \ldots$$

and conditional expectations $E_{M_n} : M_{n+1} \to M_n$, then

1. $M_n \cap \{e_{n+1}\}' = M_{n-1}$
2. $e_{n+1} x_n e_{n+1} = E_{M_{n-1}}(x_n) e_{n+1} \ \forall \ x_n \in M_n$
3. $E_{M_n}(e_{n+1}) = \tau$, or equivalently $tr(x_n e_{n+1}) = \tau tr(x_n) \ \forall x_n \in M_n$.

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Since $e_k \in M_k$, it follows from the preceding facts that

1. e_i and e_j commute for $j \geq i + 2$
2. $e_{n+1} e_n e_{n+1} = \tau e_{n+1}$
3. $tr(xe_n) = \tau tr(xe_n) \ \forall \ x \in M_{n-1}$ (and in particular, if x is a 'word in $e_1, \cdots, e_{n-1}$'

An unexpected and pleasant offshoot of Jones' work on subfactors was the celebrated connection with knot theory. The initial contact seems to have been made when Pierre de la Harpe, a collaborator of Jones, remarked on the striking similarity between the relations satisfied by the Jones projections on the one hand, and the so-called Braid relations on the other.

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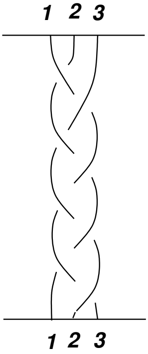
A braid is what you think it is:



$$(= (b_1^{-1} b_2)^k)$$

3–strand braid

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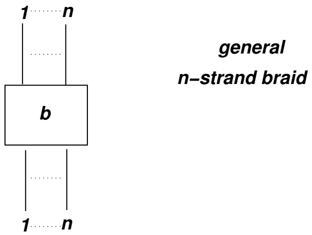
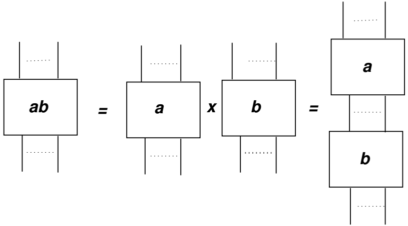


where the 'black box b' contains all the knotting/braiding between the different strands.

The collection B_n of all n-strand braids turns out to be a group with respect to the multiplication defined by:



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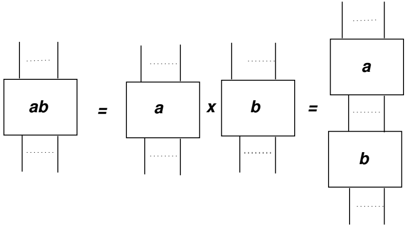

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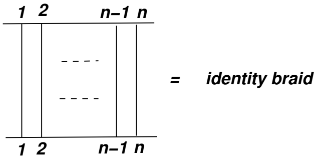
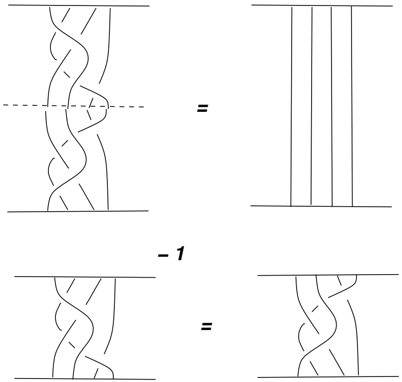


The inverse of a braid is obtained by reflecting in a horizontal mirror placed at the level of the lower frame of the braid: for example,



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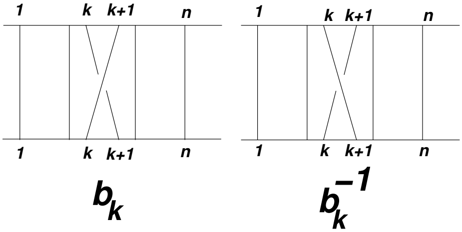
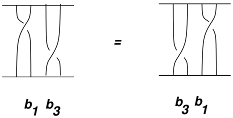


Tne braid relations

The b_j's satisfy the following relations:

$b_i b_j = b_j b_i$ if $|i - j| \geq 2$



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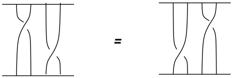
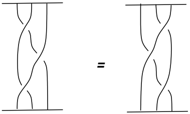


$b_1 \; b_3 \qquad b_3 \; b_1$

$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$ for all $i < n - 1$



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where

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This means that if any group G contains elements $g_1, g_2, \cdots g_{n-1}$ which satisfy the relations (r_1) and (r_2), then there exists a unique homomorphism $\pi : B_n \to G$ such that $\pi(b_i) = g_i \ \forall i$.

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Corollary: There exist homomorphisms $\pi : B_n \to B_{n+1}$ such that $\pi(b_i^{(n)}) = b_i^{(n+1)} \ \forall \ 1 \leq i < n$.

From braids to knots

The **closure** of a braid $b \in B_n$ is obtained by sticking together the strings connected to the j-th pegs at the top and bottom. The result is a **link**,[1] or a many component knot $\hat{b}$.

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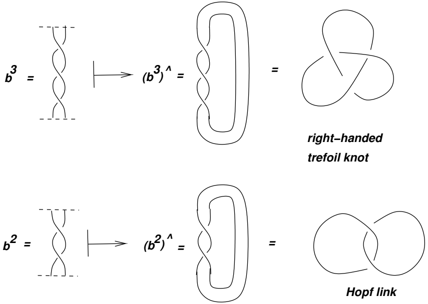

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Theorem: (Markov):

Two braids have isotopic closures iff you can pass from one to the other by a finite sequence of moves of one of two types (the so-called Markov moves of types I and II) which we now describe.

Type I Markov move:

$$c^{(n)}b^{(n)}(c^{(n)})^{-1} \leftrightarrow b^{(n)}$$

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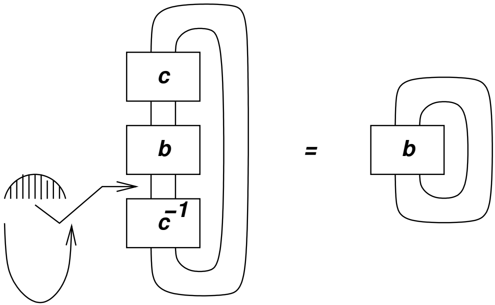

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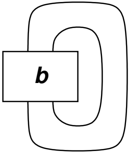

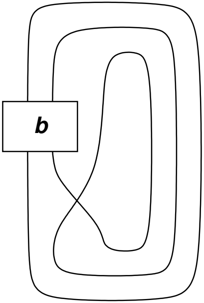

The strategy for obtaining a link invariant

Let $\mathcal{L}$ denote the set of '(tame) link diagrams'. Let $\mathcal{L} \ni L \stackrel{P}{\mapsto} P_L \in \mathcal{S}$ be any map from $\mathcal{L}$ into some set $\mathcal{S}$. Then, we may conclude the following assertion from Alexander's and Markov's theorems:

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There exist functions $P_n : B_n \to \mathcal{S}$ such that

1. $P_n(b) = P_n(cbc^{-1})\ \forall b, c \in B_n$ - i.e., each P_n is a class function (meaning it is constant on conjugacy classes).
2. $P_n(b^{(n)}) = P_{n+1}(b^{(n+1)}(b_n^{(n+1)})^{\pm 1})$
3. $P_{\hat{b}} = P_n(b)\ \forall b \in B_n$

Recall that class functions on finite groups are given by characters (= traces of representations). But the braid groups are not finite; so we seek representations taking values in II_1 factors. Thus condition (1) of the Proposition will be satisfied if we define

$$P_n(b) = tr_M(\pi_n(b))$$

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Theorem: (Jones) Suppose $N \subset M$ is a subfactor with $[M : N] = \tau^{-1} < \infty$ and associated basic construction tower

$$N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 \subset \cdots$$

Then

- there exists a unique representation $\pi_n : B_n \to GL(M_n)$ such that

$$\pi_n(b_i^{(n)}) = q^{\frac{1}{2}} \left((q+1)e_i - 1\right) \ \forall 1 \leq i < n \ ,$$

  where $q + q^{-1} + 2 = \tau^{-1}$.
- If we define

$$P_n(b) = \left\{-(q^{\frac{1}{2}} + q^{-\frac{1}{2}})\right\}^{n-1} tr_{M_n}(\pi_n(b)) \ ,$$

  then the P_n's satisfy conditions (1) and (2) of Proposition (strategy).

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Definition: A triple (L_+, L_-, L_0) of link diagrams is said to be **skein-related** if there exists (some n and) $a, b \in B_n$ and $i \in \{1, \cdots, n-1\}$ such that

$$\begin{aligned} L_+ &= \widehat{ab_i b} \\ L_- &= \widehat{ab_i^{-1} b} \\ L_0 &= \widehat{ab} \end{aligned}$$

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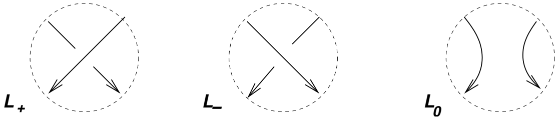
A more intuitively appealing way of saying this is that there exists one crossing away from which all three diagrams are identical, and that at that crossing, they look as follows:



Theorem: **(Jones)**

1. $V_L(q)$ is a Laurent polynomial in $q^{\frac{1}{2}}$;
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3. $V_L(q)$ is $q^{\frac{1}{2}} \times$ (a Laurent polynomial in q) if L has an even number of components;
4. The assignment $L \mapsto V_L$ is uniquely determined by the properties
   1. $$V_{U_1}{}^{2}(q) = 1$$

      and
   2. $$q^{-1} V_{L_+}(q) - q V_{L_-}(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) V_{L_0}(q)$$

   for any skein-related triple (L_+, L_-, L_0).

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The proof relies on the following fact, which in turn, is proved by an interesting kind of induction:

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The proof relies on the following fact, which in turn, is proved by an interesting kind of induction:

Lemma: Every link diagram can be transformed, by making 'sign-changes' at a suitably chosen set of crossings, into one representing an **unlink**[3].

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Define the *knottiness* $\kappa(L)$ of a link diagram L to be (n, k), where

- n is the number of crossings of L; and
- k is the minimum number of sign-changes needed to transform L to an unlink.

Our proofs by 'knotty induction' rely on the fact that the set $\mathbb{Z}_+ \times \mathbb{Z}_+$ is a *totally ordered* set (i.e., non-empty subsets have smallest elements) with respect to lexicographic ordering:

$$(n_1, k_1) \leq (n_2, k_2) \Leftrightarrow \begin{cases} n_1 < n_2 \\ \text{or} \\ n_1 = n_2 \text{ and } k_1 \leq k_2 \end{cases}$$

A knotty induction

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Hence, to prove an assertion about all link diagrams by induction on their knottiness, it suffices to

- verify it for unlinks (level 0)
- verify it for an L under the assumption that it is true for every L' with $\kappa(L') < \kappa(L)$.

Proof of Jones' theorem

We first prove that V_L satisfies the two properties listed in (4). Part (1) of (4) follows from the definition; as for part (2) of (4), observe that $g_i = \pi_n(b_i)$ has eigenvalues $q^{\frac{3}{2}}$ and $-q^{\frac{1}{2}}$, and hence satisfies the quadratic equation

$$(g_i - q^{\frac{3}{2}})(g_i + q^{\frac{1}{2}}) = 0 \ ,$$

which may be re-written as

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Next we simultaneously establish assertions (2) and (3) of the theorem by our method of knotty induction. To set the ball rolling, observe that putting $b = 1^{(n)}$ in the definition of P_n, we get

$$V_{U_n}(q) = \left\{-(q^{\frac{1}{2}} + q^{-\frac{1}{2}})\right\}^{n-1}$$

so the unlinks do satisfy (2) and (3).

Suppose next that $L \in \mathcal{L}$ is not an unlink, and that $L' \in \mathcal{L}$ satisfies (2) and (3) whenever $\kappa(L') < \kappa(L) = (n, k)$ (say). Then $n > 0$ and $k > 0$ (since L is not an unlink) and hence there is one crossing of L such that if L' is the link obtained by changing that crossing from an over-crossing to an under-crossing (or vice versa) then $\kappa(L') = (n, k')$ with $k' < k$. Hence there exists an $L" \in \mathcal{L}$ with $\kappa(L") = (n-1, k")$ such that $(L, L', L")$ (or $(L', L, L")$) is a skein-related triple. The fact that $\kappa(L) > \max\{\kappa(L'), \kappa(L")\}$ and the already established fact that V_L satisfies the skein relation (4)(2) is now seen to imply that L must also satisfy (2) and (3).

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For the last assertion of the previous paragraph, one also needs to observe that L_+ and L_- always have the same number of components while L_0 has one more (or less) component than L_0.

Finally, it is a pleasant exercise in this knotty induction to show that if P_L is any knot invariant satisfying parts (1) and (2) of (4), then we must have $P_L = V_L$.

An example

We conclude by listing some exercises which demonstrate the method of computation of V_L for a desired L (where we consider the example of $L = T_+$, the so-called right-handed trefoil. For this, we first identify a few links.

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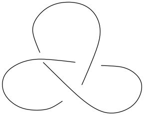

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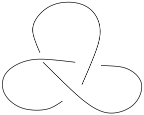
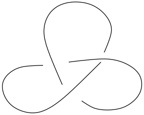


left–handed trefoil

T_-



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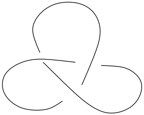

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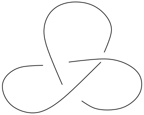
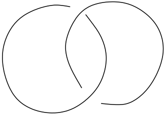
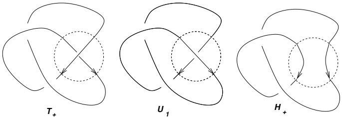


Hopf link

H_+



Suppose we want to compute V_{T_+}. We look at one crossing of T_+, and observe that (T_+, U_1, H_+) is a skein-related triple



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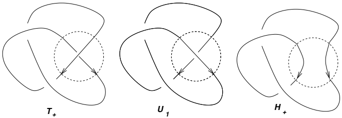
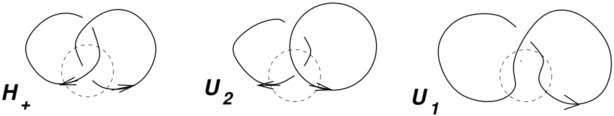


so the skein relation reduces the problem to computing V_{H_+}.

Next, look at a crossing of H_+ and note that (H_+, U_2, U_1) is a skein-related triple



so the skein relation tells us how to compute V_{H_+}, since we know that

$$V_{U_n}(q) = \left\{-(q^{\frac{1}{2}} + q^{-\frac{1}{2}})\right\}^{n-1}$$

Explicitly, our computations reveal that

$$q^{-1}V_{H_+}(q) - qV_{U_2}(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})V_{U_1}(q)$$

and hence,

$$\begin{aligned}
V_{H_+}(q) &= q\left(qV_{U_2}(q) + (\sqrt{q} - \frac{1}{\sqrt{q}})V_{U_1}(q)\right) \\
&= q\left(-\frac{q(q+1)}{\sqrt{q}}) + \frac{q-1}{\sqrt{q}}\right) \\
&= -\sqrt{q}(q^2+1)
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and hence,

$$\begin{aligned} V_{T_+}(q) &= q\left(qV_{U_1}(q) + (\sqrt{q} - \frac{1}{\sqrt{q}})V_{H_+}(q)\right) \\ &= \cdots \\ &= q + q^3 - q^4 \end{aligned}$$

Analogously, it can also be shown that

$$V_{T_-}(q) = q^{-1} + q^{-3} - q^{-4}$$

But more generally, we have the following fact, also provable easily by our method of knotty induction:

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In conclusion, the Jones polynomial allows us to make the topologically non-trivial statment that the links T_+, T_-, H_+, U_n are all pairwise inequivalent.

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