# Operator algebras - stage for non-commutativity
## (Panorama Lectures Series)
## IV. $II_1$ factors and their subfactors

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## $II_1$ factors

Recall that a von Neumann algebra (vNa) is called a factor if $Z(M) = M \cap M' = \mathbb{C}$; and that a factor $M$ is said to be *finite* if

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The following conditions on two projections $p, q$ in a finite factor $M$, are equivalent:

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Let $M$ be a finite factor. There are two possibilities:

1. $dim_{\mathbb{C}} M < \infty$. In this case $M \cong M_n(\mathbb{C}) = \mathcal{L}(\mathbb{C}^n)$ for a unique $n$, and $\{tr_M p : p \in \mathcal{P}(M)\} = \{\frac{k}{n} : 0 \leq k \leq n\}$.
2. $dim_{\mathbb{C}} M = \infty$. Then $M$ is a $II_1$ factor, and in this case, $\{tr_M p : p \in \mathcal{P}(M)\} = [0, 1]$.

Henceforth, $M$ will be a $II_1$ factor.

*Def:* An **$M$-module** is a separable Hilbert space $\mathcal{H}$, equipped with a morphism $\pi : M \to \mathcal{L}(\mathcal{H})$ of von Neumann algebras (i.e., a normal representation). Two $M$-modules are isomorphic if there exists an invertible (equivalently, unitary) $M$-linear map between them.

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*Proposition:* There exists a complete isomorphism invariant

$$\mathcal{H} \mapsto \mathbf{dim_M}\mathcal{H} \in [0, \infty]$$

of $M$-modules such that:

- $\mathcal{H} \cong \mathcal{K} \Leftrightarrow dim_M\mathcal{H} = dim_M\mathcal{K}$.
- $dim_M(\oplus_n \mathcal{H}_n) = \sum_n dim_M\mathcal{H}_n$.
- For each $d \in [0, \infty]$, $\exists$ an $M$-module $\mathcal{H}_d$ with $dim_M\mathcal{H}_d = d$.

In view of the uniqueness of $tr_M$, we shall simply write $L^2(M)(= (\hat{M})^{\urcorner}$. It is true as in the finite dimensional case that there exist the *left* and *right regular representations* of $M$ on $L^2(M)$ which satisfy

- $\lambda_M(x)\hat{y} = \widehat{xy} = \rho_M(y)\hat{x} \ \forall x, y \in M$; and
- $(\lambda_M(M))' = \rho_M(M)''$

As before, we identify $x \in M$ with $\lambda_M(x) \in \mathcal{L}(L^2(M))$.

# The standard form of a $II_1$ factor

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It follows that $K_0(M) \cong \mathbb{R}$.

**The hyperfinite $II_1$ factor $R$:** Among $II_1$ factors, pride of place goes to the ubiquitous hyperfinite $II_1$ factor $R$. It is characterised as the unique $II_1$ factor which has any of several properties, such as injectivity and approximate finite-dimensionality ($=$ hyperfiniteness).

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Thus, up to isomorphism, there exists a unique $II_1$ factor $R$ which contains an increasing sequence

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*Examples of $II_1$ factors:* Let $\lambda : G \to \mathcal{U}(\mathcal{L}(\ell^2(G)))$ denote the left-regular representation of a countable infinite group $G$, and let $LG = (\lambda(G))''$. **The group von Neumann algebra $LG$** is a $II_1$ factor iff every conjugacy class of $G$ other than $\{1\}$ is infinite.

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*Big open problem:* is $L\mathbb{F}_2 \cong L\mathbb{F}_3$? (Compare with the $C^*_{red}$ case.)

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A **subfactor** is a unital inclusion $N \subset M$ of $II_1$ factors. For a subfactor as above, Jones defined the **index of the subfactor** by

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A subfactor $N$ is said to be **irreducible** if $N' \cap M = \mathbb{C}$ - or equivalently, if $L^2(M, tr_M)$ is irreducible as an $N - M$ bimodule - meaning it has no non-zero submodule other than itself.

It is known that if a subfactor $N \subset M$ has finite index, then $N$ is hyperfinite if and only if $M$ is. In this case, call the subfactor hyperfinite.

# Possible values of index of irreducible subfactors

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Very little is known about the set $\mathcal{I}_R^0$ of possible index values of irreducible hyperfinite subfactors.

1. (Jones) $\mathcal{I}_R = [4, \infty] \cup \{4cos^2(\frac{\pi}{n}) : n \geq 3\}$ and $\mathcal{I}_R^0 \supset \{4cos^2(\frac{\pi}{n} : n \geq 3\}$
2. $\left(\frac{N+\sqrt{N^2+4}}{2}\right)^2, \left(\frac{N+\sqrt{N^2+8}}{2}\right)^2 \in \mathcal{I}_R^0 \ \forall N \geq 1$
3. $(N + \frac{1}{N})^2$ is the limit of an increasing sequence in $\mathcal{I}_R^0$.

# Automorphisms

We list below a few facts concerning **automorphisms** of von Neumann algebras:

1. If $\pi : M \to N$ is a normal homomorphism of von Neumann algebras, there exists a central projection $z$ such that $ker\ \pi = Mz = \{xz : x \in M\}$.
2. If $\pi$ is a $*$-isomorphism of von Neumann algebras (just algebraically *à priori*), then it is automatically normal.
3. if $\pi : M \to N$ is a $*$-homomorphism of a factor onto a von Neumann algebra, then $\pi$ is identically zero or a normal isomorphism.
4. Thus an algebraic $*$-automorphism of a von Neumann algebra is automatically normal.
5. An automorphism of a finite factor $M$ preserves $tr_M$.
6. An automorphism $\theta$ of $M$ is said to be *free* if

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**Proposition:**

1. Suppose $M = L^\infty(X, \mathcal{B}, \mu)$, with $\mu$ $\sigma$-finite. Then
   1. $\theta \in Aut(M) \Leftrightarrow$ there exists a non-singular automorphism $T$ of $(X, \mathcal{B}, \mu)$ such that $\theta(f) = f \circ T^{-1}$.
   2. $\theta \in Aut(M)$ is free iff it moves almost all points - i.e., $\mu(\{x \in X : Tx = x\}) = 0$.
2. An automorpism of a factor is free iff it is outer - i.e., it is not inner, meaning there is no $u \in \mathcal{U}(M)$ such that $\theta(x) = uxu^* \ \forall x \in M$

*Definitions:*

1. An **action** of a group $G$ on a von Neumann algebra $M$ (written $G \curvearrowright M$) is a group homomorphism $\alpha$ from $G$ into the group $Aut(M)$ of $*$-automorphisms of $M$.
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**Proposition:**

1. For any $n$, $U_n(\mathbb{C}) = \mathcal{U}(M_n(\mathbb{C}))$ - and hence every finite group - admits an outer action on $R$.
2. If $G \curvearrowright R$ is an outer action of a finite group $G$ on $R$, the fixed subalgebra $R^G = \{x \in R : g \cdot x = x \forall g \in G\}$ is a subfactor of $R$ with $[R : R^G] = |G|$.
3. If $G \curvearrowright R$ is as in (2) above, then every intermediate $*$-subalgebra $R^G \subset P \subset R$ is of the form $P = R^H$ for some subgroup $H$ of $G$; further, $[R^H : R^G] = [G : H]$.
4. If $G_i \curvearrowright R$, $i = 1, 2$ are outer actions of finite groups, then $(R^{G_1} \subset R) \cong (R^{G_2} \subset R) \Leftrightarrow G_1 \cong G_2$.

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**Proposition:** Suppose $N \subset M$ is a subfactor. Then $L^2(N)$ sits naturally as a subspace of $L^2(M)$. Let us write $e_N$ for the orthogonal projection of $L^2(M)$ onto $L^2(N)$.

1. Then $e_N(\hat{M}) \subset \hat{N}$, and we define $E_N$, the so-called *tr-preserving conditional expectation of M onto N* by

$$\widehat{E_N(m)} = e_N(\hat{m})$$

2. The map $E_N$ satisfies and is characterised by the following properties:
   - $tr|_N = tr \circ E$.
   - $E(nm) = nE(m)$, i.e., $E_N$ is $N$-linear.
3. $e_n m e_n = E(m) e_N$, where, as usual, we identify $m \in M$ with $\lambda_M(m)$.

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The **modular conjugation associated to M** is the antiunitary operator $J_M$ defined on $L^2(M)$ by $J_M(\hat{x}) = \widehat{x^*}$.

**Proposition:** For a subfactor $N \subset M$, simply writing $J$ for $J_M$ and $e$ for $e_N$, we have:

- $JxJ = \rho_M(x^*) \; \forall x \in M$
- $Je = eJ$
- $JN'J = (M \cup \{e\})''$, where $N'$ means $\lambda_M(N)'$ in $\mathcal{L}(L^2(M))$
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**Proposition:**
If $[M : N] < \infty$, then

1. $N' \cap M$ is finite-dimensional; in fact, $dim(N' \cap M) \leq [M : N]$; and
$$[M : N] < 4 \Rightarrow N' \cap M = \mathbb{C}.$$
2. $M_1 =: \ < M, e >= (M \cup \{e\})''$ is also a $II_1$ factor and $[M_1 : M] = [M : N]$.
3. $E_M(e) = \frac{1}{[M:N]} \ 1$

If $N \subset M$ is a finite index subfactor, we write

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# Basic construction tower

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Since $M \subset M_1$ is also a finite index subfactor, we can play the game once more, and in fact *ad infinitum (nauseum?)*, to get a tower

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Since the index is multiplicative, we see that $[M_i : M_j] = [M : N]^{j-i}$.

Thus we have the following grid of finite-dimensional $C^*$-algebras:

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\mathbb{C} & = & N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & \cdots \\
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Further, this comes equipped with a consistent trace (which, on $M_i' \cap M_j$ is the restriction of $tr_{M_j}$). This grid, with this trace, is called **the standard invariant** of $N \subset M$.

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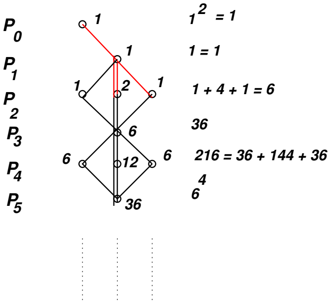
This turns out to be a complete invariant for a 'good class' of subfactors - the so-called **extremal** ones.

# An example

To better understand this standard invariant, start by observing that the tower in the first row of the grid is described by the total Bratteli diagram obtained by glueing the several individual Bratteli diagrams together. We illustrate varous features of this tower in the example $R^{S_3} \subset R$:




Here, we have written $P_k = N' \cap M_{k-1}$. The diagram illustrates several features that are present in the corresponding diagram of relative commutants for every subfactor:

(a) The part of the diagram between the $n$th and $(n+1)$-st floors consists of two parts: (i) a (horizontal) mirror-reflection of the part of the diagram between the $(n-1)$-th and $n$th floors, and (ii) a 'new part'. In fact, new vertices, if any, on the $(n+1)$-st floor are connected *only* to new vertices on the $n$-th floor.

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(b) The (red) graph comprising all the 'new parts' is called the **principal graph** $\Gamma$ of the subfactor $N \subset M$. (It follows from (a) that the Bratteli diagram for the entire tower $\{N' \cap M_{k-1} : k \geq 0\}$ is determined by the principal graph.)

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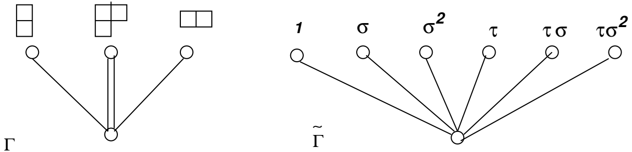
(d) In the exhibited example, the principal graph and the dual principal graph are given by:



(e) It is a fact that $\Gamma$ is finite iff $\widetilde{\Gamma}$ is finite, in which case the subfactor is said to have **finite depth**.

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In addition to the two principal graphs, which only describe the two towers of relative commutants, one also needs to encode the data of how one tower is embedded into the next. This has been done in at least three ways: as a **paragroup** (Ocneanu), a **$\lambda$-lattice** (Popa), or a **planar algebra** (Jones). Any one of these notions is equivalent to the ‘standard invariant, and is a complete invariant, provided the subfactor is **extremal**. (Finite depth subfactors are known to be extremal, and thus determined by their standard invariant.)

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We shall content ourselves with recording the following relations satisfied by the Jones projections $\{e_n : n \geq 1\}$ (which are easy consequences of the basic construction):

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We shall content ourselves with recording the following relations satisfied by the Jones projections $\{e_n : n \geq 1\}$ (which are easy consequences of the basic construction):

$$\begin{array}{rcll} e_i^2 & = & e_i & \forall i \\ e_i e_j & = & e_j e_i & \text{if } |i-j| \geq 2 \\ e_i e_j e_i & = & \tau e_i & \text{if } |i-j| = 1 \end{array}$$

where $\tau = [M : N]^{-1}$.

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