The GJS construction for graphs. I: Presentation of two categories

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Overview

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2. Outline of proof
    - Weighted bipartite graphs
    - The graded ($Gr(\Gamma)$) and filtered ($F(\Gamma)$) versions of the algebras
    - The factor $M(\Gamma)$

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   - The factor $M(\Gamma)$
3. Two categories and their presentations

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Our proof entails what may be considered the GJS construction associated to a finite graph with its Perron-Frobenius weighting. The next few slides attempt to describe this construction.

For us, a graph will be a triple $\Gamma = (V, E, s, r, \tilde{})$, with

- V is a finite set of vertices
- $E \subset V \times V$ is a set of edges
- $s, r : E \to V$ are the 'range' and 'source' maps, and
- $E \ni e \to \tilde{e} \in E$ denotes 'orientation reversal' (so that $r(e) = s(\tilde{e})$)

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A *weighting* on Γ is a function $\mu : V \to (0, \infty)$. We will primarily be conceened with the case when μ^2 is an eigenvector for the adjacency matrix of Γ: i.e., there exists a positive number $\delta > 0$ such that

$$\sum_{e \in E : s(e) = v} \mu(r(e))^2 = \delta \mu(v)^2 \ \forall\ v \in V.$$

However, in the course of proving the theorem, we will need to consider graphs where the weighting is not the Perron-Frobenius one.

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$$Gr(\Gamma) = \oplus_{n \geq 0} P_n(\Gamma) \ ,$$

with multiplication, denoted by $\bullet$, given on basis elements by concatenation of paths when defined, thus:

$$[\xi] \bullet [\eta] = \begin{cases} [\xi \circ \eta] & \text{if } r(\xi) = s(\eta) \\ 0 & \text{if } r(\xi) \neq s(\eta) \end{cases}$$

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$Gr(\Gamma)$ admits a faithful positive tracial state τ whose slightly complicated definition we omit.

There is a filtered involutive algebra $F(\Gamma)$ which is also $\oplus_{n\geq 0} P_n(\Gamma)$ as a vector space, with involution defind exactly as in $Gr(\Gamma)$. This $F(\Gamma)$ has the advantage over $Gr(\Gamma)$ that it has a faithful positive tracial state t with the following simple definition:

$$t([\xi]) = \begin{cases} \mu^2(v) & \text{if } \xi = v \text{ has length } 0 \\ 0 & \text{if } \xi \text{ has length} > 0 \end{cases}$$

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The trade-off is that the multiplication in $F(\Gamma)$, denoted $\#$, is more complicated: in fact

$$[\xi]\#[\eta] = \sum_{k=0}^{min\{m,n\}} ([\xi]\#[\eta])_{m+n-2k}$$

with $([\xi]\#[\eta])_t \in P_t(\Gamma)$.

Theorem

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3. *If* Γ *is connected and has at least two edges, and is equipped with the Perron-Frobenius weighting, then* $M(\Gamma) = \lambda(A)''$ *is an interpolated free group factor with finite parameter.*

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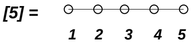

A typical element of $Hom([m],[n])$ consists of a rectangle with m points in the bottom row, n points in the top row, and a Temperley-Lieb diagram within this rectangle such that every string terminates in one of these $m+n$ points, with every point on the top row being connected to a point on the bottom row.

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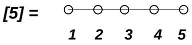

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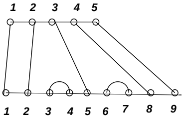

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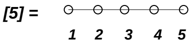

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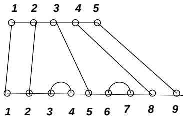

The notation $\mathcal{E}$ is meant to signify *Epi*. Also, it should be clear that $Hom([m],[n]) \neq \emptyset$ iff $m-n$ is non-negative and even.

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For instance, the element of $Hom([9], [5])$ illustrated on the previous slide can be written as product (=the composite) $S_3^7 S_6^9$. If we 'read the caps in the opposite order' we find that we can also write the same morphism as the product $S_4^7 S_3^9$. A little thought shows that we always have

$$S_p^{n-2} S_q^n = S_q^{n-2} S_{p+2}^n \ \forall \ n-2 > p \geq q \geq 1 \ . \tag{0.1}$$

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In fact, the category $\mathcal{E}$ is generated by the S_i^n's, meaning that every morphism not in $\{id_{[n]} : n \geq 0\}$ is expressible as a product of these generators. (*Reason:* A morphism $S \in Hom([m+2k],[m])$ is uniquely expressible as $S = S_{i_1}^{m+2} S_{i_2}^{m+4} \cdots S_{i_k}^{m+2k}$ with $1 \leq i_1 < i_2 < \cdots < i_k$. In fact, S has this representation iff the left ends of its caps at $i_1, \cdots, i_k$.

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Given any product of these generators, we may repeatedly use (0.1) to replace it by one where the subscripts are strictly increasing.

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Consider the maps $\tilde{S}_i^n : P_n(\Gamma) \to P_{n-2}(\Gamma), 1 \le i < n$ defined by

$$\tilde{S}_i^n([\xi]) = \delta_{\xi_i, \tilde{\xi}_{i+1}} \frac{\mu(v_i)}{\mu(v_{i\pm1})} [\xi_{[0,i-1]} \circ \xi_{[i+1,n]}] \ ,$$

where we write ξ for the path of length n given by

$$v_0 \xi_1 v_1 \cdots v_{n-1} \xi_n v_n$$

and $\xi_{[p,q]}$ (when $p \le q$) for the obvious sub-path

$$v_p \xi_{p+1} v_{p+2} \cdots v_{q-1} \xi_q v_q \ .$$

These are seen (assuming PF-weighting) to satisfy the relations 0.1, and to consequently yield an 'action of $\mathcal{E}$'.

The multiplication in $F(\Gamma)$

In terms of the maps $\tilde{S}_i^n$ of last page, the multiplication in $F(\Gamma)$ may be explicitly described, thus: if $[\xi] \in P_m(\Gamma)$ and $[\eta] \in P_n(\Gamma)$, then

$$[\xi]\#[\eta] = \sum_{k=0}^{min\{m,n\}} ([\xi]\#[\eta])_{m+n-2k} ,$$

where

$$([\xi]\#[\eta])_{m+n-2k} = \begin{cases} [\xi \circ \eta] & \text{if } k = 0 \\ S_{m-k+1}^{m+n-2(k-1)} \cdots S_{m-1}^{m+n-2} S_m^{m+n}([\xi \circ \eta]) & \text{if } k > 0 \end{cases}$$

This category depends on a parameter $\delta > 0$, and its objects are also denoted by $[n], n \geq 0$. Although we use the same symbol, we shall think of $[n]$ as a set of $2n$ points listed in a line (exactly as the object of $\mathcal{E}$ that is denoted by $[2n]$).

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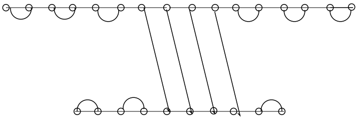
A typical element of $Hom([m],[n])$ is denoted by a symbol such as $T(P,Q)^n_m$ where $P \subset [n]$ and $Q \subset [m]$ are intervals of even cardinality. The morphism $T([4,5],[3.4])^8_5$ is illustrated below:



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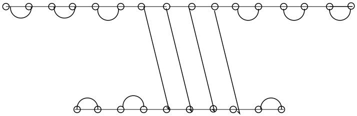


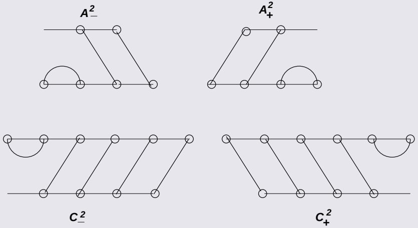
A moment's thought shows that, in general,

$$T(P, Q)^m_n \circ T(R, S)^n_p = \delta^{n-|Q \cup R|} T(Y, Z)^m_p$$

for appropriate Y, Z, m, p.

Theorem

The category $\mathcal{C}(\delta)$ is generated by the collection of those morphisms which have exactrly one cup or exactly one cap, i.e., the collection $\{A^n_\pm : n \geq 1\} \cup \{C^n_\pm : n \geq 0\}$ where the superscript denotes the domain and the letters A and C are meant to indicate annihilation and creation. We illustrate these morphisms below for $n = 2$:



Theorem

The category is presented by the preceding set of generators and the relations:

$$A^1_- = A^1_+ \tag{0.2}$$

$$C^0_- = C^0_+ \tag{0.3}$$

$$A^{n+1}_- A^{n+2}_+ = A^{n+1}_+ A^{n+2}_- \tag{0.4}$$

$$A^{n+1}_- C^n_- = \delta \ id_{[n]} \tag{0.5}$$

$$A^{n+2}_- C^{n+1}_+ = C^n_+ A^{n+1}_- \tag{0.6}$$

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For example, the morphism $T([4,5],[3.4])^8_5$ illustrated earlier has 'canonical form' given by

$$T([4,5],[3.4])^8_5 = C^7_+ C^6_+ C^5_+ C^4_- C^3_- C^2_- A^3_+ A^4_- A^5_- \ ,$$

as can be seen by 'listing all the caps from left to right and then all cups from left to right'.

It is tempting to mention that we believe we can prove, although we have not yet dotted the 'i's and crossed the 't's, that our main theorem remains valid for any connected graph with at least two edges, and with any weighting, with possible loops and multi-edges.

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The pleasant little off-shoot of this general 'result' would be that if Γ is the flower with n petals (i.e., one vertex and n edges), the associated $M(\Gamma)$ is, to no one's surprise, just $L\mathbb{F}(n)$.