Double Cones are Intervals

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Abstract

The purpose of this short note is to point out the following fact and some of its pleasant 'consequences': the so-called double-cones in (4-dimensional) Minkowski space are nothing but the intervals $(A, B) = \{C \in H_2 : (C - A)$ and $(B - C)$ are both positive-definite$\}$ in the space H_2 of 2×2 complex Hermitian matrices.

1 Introduction

This short note is a result of two events: (i) I was recently approached by the editors of a volume being brought out in the memory of Paul Halmos, and I certainly wanted to contribute some token of many fond memories of Paul; and (ii) some time ago, I learnt something with considerable pleasure which, I am sure, is just the kind of 'fun and games with matrices' that brought a gleam into Paul's eye. The one other reason for presuming to think that this note might interest other people is that when I communicated this 'discovery' to some of my colleagues - a harmonic analyst and a physicist, each with some 30 years exposure to the standard facts about Minkowski metrics, etc. - I was greeted with a response along the lines of : 'How pretty'!

2 The $(H_2, |\cdot|)$ model of Minkowski space

It must be stated that most of what follows is probably 'old hat' and the many things put down here are for the sake of setting up notation preparatory to justifying the assertion of the abstract.

In the language of the first pages of a physics text on relativity, Minkowski space is nothing but 4 $(= 1 + 3)$-dimensional real space

$\mathbb{R}^4 = \{x = (x_0, x_1, x_2, x_3) : x_i \in \mathbb{R}\ \forall i\}$ equipped with the form defined by $q(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2$. We shall prefer to work with a 'matricial' model (which might appeal more to an operator-theorist).

Thus we wish to consider the real Hilbert space

$$H_2 = \{\left(\begin{array}{cc} a & z \\ \bar{z} & b \end{array}\right) : a, b \in \mathbb{R}, z \in \mathbb{C}\}$$

of 2×2 complex Hermitian matrices, and observe that the assignment

$$\mathbb{R}^4 \ni x = (x_0, x_1, x_2, x_3) \stackrel{\phi}{\mapsto} X = \left(\begin{array}{cc} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{array}\right) \in H_2 \tag{2.1}$$

defines a (real-)linear isomorphism, and that we have the following identities:

$$q(x) = |X|\ ,\ x_0 = tr(X)$$

where $|X|$ denotes the determinant of the matrix X and $tr(X) = \frac{1}{2}Tr(X)$ denotes the normalised trace (= the familiar matrix trace scaled so as to assign the value 1 to the identity matrix I).

It should be noted that the isomorphism ϕ of equation (2.1) can be alternatively written as

$$\phi(x) = \sum_{i=0}^{3} x_i \sigma_i$$

where σ_0 is the identity matrix, I and $\sigma_1, \sigma_2, \sigma_3$ are the celebrated *Pauli matrices* and that $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ is an orthonormal basis for H_2 with respect to the normalised Hilbert- Schmidt inner product given by $\langle X, Y\rangle = tr(Y^*X)$, so ϕ is even a (real) unitary isomorphism.

Recall that the so-called *positive* and *negative light cones* (at the origin) are defined as

$$\begin{array}{rcl} C^+(0) & = & \{x \in \mathbb{R}^4 : x_0 > 0, q(x) > 0\} \\ C^-(0) & = & \{x \in \mathbb{R}^4 : x_0 < 0, q(x) > 0\} \\ & = & -C^+(0) \end{array}$$

while the 'positive light cone' and the 'negative light cone' at $x \in \mathbb{R}^4$ are defined as

$$\begin{array}{rcl} C^+(x) & = & \{y \in \mathbb{R}^4 : (y_0 - x_0) > 0, q(y - x) > 0\} \\ & = & x + C^+(0) \\ C^-(x) & = & \{y \in \mathbb{R}^4 : (y_0 - x_0) < 0, q(y - x) > 0\} \\ & = & x - C^+(0)\ . \end{array}$$

Finally, a *double-cone* is a set of the form $D(x, y) = C^+(x) \cap C^-(y)$ (which is non-empty precisely when $y \in C^+(x)$).

The key observation for us is the following

REMARK 2.1.

$$\phi(C^+(0)) = P \tag{2.2}$$

where P is the subset of H_2 consisting of positive-definite *matrices.*

Reason: Any $X \in H_2$ has two real eigenvalues $\lambda^*(X)$ and $\lambda_*(X)$ satisfying $\lambda_*(X) \leq \lambda^*(X)$ and

$$\frac{1}{2}(\lambda_*(X) + \lambda^*(X)) = tr(X),\ \lambda_*(X) \cdot \lambda^*(X) = |X|$$

so

$$X \in P \Leftrightarrow \lambda^*(X), \lambda_*(X) > 0 \Leftrightarrow tr(X), |X| > 0.$$

Thus we do indeed find that

$$\begin{aligned} C^+(X) &= X + P \\ C^-(Y) &= Y - P \\ D(X, Y) &= (X, Y) = \{Z \in H_2 : X < Z < Y\}\ , \end{aligned}$$

where we write $A < C \Leftrightarrow C - A \in P$. (It should be emphasised that $A < C$ implies that $C - A$ is invertible, not merely positive semi-definite.) □

It should be observed that the relative compactness of these double cones is an easy corollary of the above remark. (I am told that this fact is of some physical significance.)

3 Some Applications

This section derives some known facts using our Remark 2.1.

PROPOSITION 3.1. *The collection $\{D(x, y) : y - x \in C^+(0)\}$ (resp., $\{(X, Y) : Y - X \in P\}$) forms a base for the topology of $\mathbb{R}^4$ (resp., H_2).*

Proof: Suppose U is an open neighbourhood of a $Z \in H_2$. By definition, we can find $\epsilon > 0$ such that $W \in H_2, \|W - Z\| < 2\epsilon \Rightarrow W \in U$, where we write $\|A\| = \max\{\|Av\| : v \in \mathbb{C}^2, \|v\| = 1\}$. Then $X = Z - \epsilon I, Y = Z + \epsilon I$ satisfy $Z \in (X, Y) \subset U$. $\square$

Thus the usual topology on $\mathbb{R}^4$ has a basis consisting of 'intervals'.

Recall next that the *causal complement* $O^\perp$ of a set $O \subset \mathbb{R}^4$ (resp., H_2) is defined by

$$O^\perp = \{z \in \mathbb{R}^4 : q(z - w) < 0 \ \forall w \in O\}$$

(resp.,

$$O^\perp = \{Z \in H_2 : |Z - W| < 0 \ \forall W \in O\}. \ )$$

PROPOSITION 3.2.

$$(X, Y)^{\perp\perp} = (X, Y) \quad \forall X < Y.$$

Proof: We shall find it convenient to write P_0 for the set of positive semi-definite matrices (i.e., $Z \in P_0 \Leftrightarrow \lambda_*(Z) \geq 0$).

We assert now that

$$Z \in (X, Y)^\perp \quad \Leftrightarrow \quad \text{there exist unit vectors } v_1, v_2 \text{ such that } \langle Zv_1, v_1 \rangle \leq \langle Xv_1, v_1 \rangle \text{ and } \langle Zv_2, v_2 \rangle \geq \langle Yv_2, v_2 \rangle \tag{3.3}$$

Notice first that, by definition,

$$Z \in (X, Y)^\perp \Leftrightarrow |Z - W| < 0 \text{ whenever } W \in (X, Y)$$

Choose $\epsilon > 0$ such that $Y - X > \epsilon I$; so also $X + \epsilon I \in (X, Y)$ and $Y - \epsilon I \in (X, Y)$. If $Z \in (X, Y)^\perp$, we find that $|Z - (X + \epsilon I)| \leq 0$ and $|Z - (Y - \epsilon I)| \leq 0$. Now a 2×2 Hermitian matrix C has negative determinant if and only if we can find unit vectors v_1 and v_2 such that $\langle Cv_1, v_1 \rangle < 0 < \langle Cv_2, v_2 \rangle$. Applying this to our situation, we can find unit vectors $v_1(\epsilon), v_2(\epsilon)$ such that $\langle Zv_1(\epsilon), v_1(\epsilon) \rangle < \langle Xv_1(\epsilon), v_1(\epsilon) \rangle + \epsilon$ and $\langle Zv_2(\epsilon), v_2(\epsilon) \rangle > \langle Yv_2(\epsilon), v_2(\epsilon) \rangle - \epsilon$. By compactness of the unit sphere in $\mathbb{C}^2$, we find, letting $\epsilon \downarrow 0$, that the condition (3.3) is indeed met.

Conversely, suppose condition (3.3) is met. If $W \in (X, Y)$, observe that $\langle Zv_1, v_1 \rangle \leq \langle Xv_1, v_1 \rangle < \langle Wv_1, v_1 \rangle$ and $\langle Zv_2, v_2 \rangle \geq \langle Yv_2, v_2 \rangle > \langle Wv_2, v_2 \rangle$ from which we may conclude that indeed $|Z - W| < 0$.

Since $O \subset O^{\perp\perp}$ $\forall O$, we only need to prove that

$$W \notin (X,Y) \Rightarrow \exists Z \in (X,Y)^{\perp} \text{ such that} |W-Z| \geq 0 \, (\Rightarrow W \notin (X,Y)^{\perp\perp}) \,.$$

If $W \notin (X,Y)$, we can find orthonormal vectors $\{v_1, v_2\}$ such that either $\langle Wv_1, v_1\rangle \leq \langle Xv_1, v_1\rangle$ or $\langle Wv_2, v_2\rangle \geq \langle Yv_2, v_2\rangle$. Suppose the former holds. (The other case is settled in the same manner.) Define the operator Z by $Z = W + N|v_2\rangle\langle v_2|$ for some N chosen so large as to ensure that $\langle Zv_2, v_2\rangle > \langle Yv_2, v_2\rangle$. We then find that also $\langle Zv_1, v_1\rangle = \langle Wv_1, v_1\rangle \leq \langle Xv_1, v_1\rangle$, and so we may deduce from condition (3.3) that $Z \in (X,Y)^{\perp}$. Since the construction ensures that $(Z - W)v_1 = 0$, we find that $|Z - W| = 0$, and the proof of the proposition is complete. □

We close with a minimal bibliography; all the background for the mathematics here can be found in [PRH], while physics-related topics such as Minkowski metric, double cones, etc., are amply treated in [BAU].

References

[PRH] Paul R. Halmos, *Finite-dimensional vector spaces*, van Nostrand, Princeton, 1958.

[BAU] H. Baumgaertel, *Operatoralgebraic Methods in Quantum Field Theory. A Set of Lectures*, (Akademie Verlag, 1995)