Superconformal field theory and operator algebras

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Quantum field theory and von Neumann algebras

Outline of the talk:

1. **$N = 1$ Super Virasoro algebras and von Neumann algebras**
   (with Carpi, Longo in 2008)
2. **$N = 1$ Super Virasoro algebras and noncommutative geometry**
   (with Carpi, Hillier, Longo in 2010)
3. **$N = 2$ Super Virasoro algebras, superstring theory and von Neumann algebras**
   (with Carpi, Longo, Xu — in progress)

The Virasoro algebra is an infinite dimensional Lie algebra generated by $\{L_n \mid n \in \mathbb{Z}\}$ and a central element c, the central charge, with the following relations.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.$$

The Lie group $\mathrm{Diff}(S^1)$ gives a Lie algebra generated by $L_n = -z^{n+1}\frac{\partial}{\partial z}$. The Virasoro algebra is a central extension of the complexification of this.

Its irreducible unitary highest weight representations have been classified and they map the central charge c to positive scalars. (This scalar is also called the central charge.)

(Some formal similarity to the Temperley-Lieb algebra.)

Super version: $N = 1$ Super Virasoro algebras

The infinite dimensional super Lie algebras generated by the central charge c, the even elements L_n, $n \in \mathbb{Z}$, and the odd elements G_r, $r \in \mathbb{Z}$ or $r \in \mathbb{Z} + 1/2$, with the following relations:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}, \\
[G_r, G_s] &= 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}.
\end{aligned}$$

The Ramond algebra, if $r \in \mathbb{Z}$

The Neveu-Schwarz algebra, if $r \in \mathbb{Z} + 1/2$

Fix a vacuum representation π of the $N = 1$ super Virasoro algebra and simply write L_n for $\pi(L_n)$.
Consider $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$, the stress-energy tensor, and $G(z) = \sum_{r\in\mathbb{Z}+1/2} G_r z^{-r-3/2}$, the super stress-energy tensor. These power series with $z \in \mathbb{C}$, $|z| = 1$ give operator-valued distributions on S^1.
Fix an interval I and take a C^∞-function f with $\operatorname{supp} f \subset I$.
We have (unbounded) operators $\langle L, f\rangle$, $\langle G, f\rangle$.
$A(I)$: the von Neumann algebra generated by these operators with various f.

The $\mathbb{Z}_2$-grading of the super Lie algebra passes to the $\mathbb{Z}_2$-grading of the operator algebras.

Quantum Field Theory: (mathematical setting)

1. Spacetime (e.g., Minkowski space)
2. Symmetry group (e.g., Poincaré group)
3. Quantum fields (operator-valued distributions on the spacetime)

Conformal Field Theory

Two-dimensional Minkowski space $\{(x,t) \mid x,t \in \mathbb{R}\}$

$\to$ One of the light rays $x = \pm t$ compactified to S^1.

Orientation preserving diffeomorphism group $\mathrm{Diff}(S^1)$.

We have operator-valued distributions acting on a Hilbert space of states having a vacuum vector.

Operator algebraic axioms: (superconformal field theory)

Motivation: Operator-valued distributions $\{T\}$ on S^1.

Fix an interval $I \subset S^1$, consider $\langle T, f\rangle$ with $\operatorname{supp} f \subset I$.

$A(I)$: the $\mathbb{Z}_2$-graded von Neumann algebra generated by these (possibly unbounded) operators

1. $I_1 \subset I_2 \Rightarrow A(I_1) \subset A(I_2)$.
2. $I_1 \cap I_2 = \varnothing \Rightarrow [A(I_1), A(I_2)] = 0$. (graded commutator)
3. $\mathrm{Diff}(S^1)$-covariance (conformal covariance)
4. Positive energy/Vacuum vector

Such a family $\{A(I)\}$ is called a superconformal net. The even part gives a local conformal net.

Each $A(I)$ is usually an injective type III_1 factor (the Araki-Woods factor). The even part of the superconformal net gives a local conformal net.

Representation theory of local conformal nets (Doplicher-Haag-Roberts):
We have a braided tensor category. Each representation is given by an endomorphism and its dimension is given by the square root of the Jones index of the image.

K-Longo-Müger: Complete rationality characterizes finiteness of the number of the irreducible representations and their finite dimensionality. ($\longrightarrow$ modular tensor category)
($\sim$ finite depth subfactors)

Geometric aspects of local conformal nets

Consider the Laplacian Δ on an n-dimensional compact oriented Riemannian manifold. Weyl formula:

$$\mathrm{Tr}(e^{-t\Delta}) \sim \frac{1}{(4\pi t)^{n/2}}(a_0 + a_1 t + \cdots),$$

where the coefficients have a geometric meaning.

The conformal Hamiltonian L_0 of a local conformal net is the generator of the rotation group of S^1.

For a nice local conformal net, we have an expansion

$$\log \mathrm{Tr}(e^{-tL_0}) \sim \frac{1}{t}(a_0 + a_1 t + \cdots),$$

where a_0, a_1, a_2 are explicitly given. (K-Longo)

This gives an analogy of the Laplacian Δ of a manifold and the conformal Hamiltonian L_0 of a local conformal net.

Noncommutative geometry:

Slogan: Noncommutative operator algebras are regarded as function algebras on noncommutative spaces.

In geometry, we need manifolds rather than compact Hausdorff spaces or measure spaces.

The Connes axiomatization of a noncommutative compact Riemannian spin manifold: spectral triple $(\mathcal{A}, H, D)$.

1. $\mathcal{A}$: $*$-subalgebra of $B(H)$, the smooth algebra $C^\infty(M)$.
2. H: a Hilbert space, the space of L^2-spinors.
3. D: an (unbounded) self-adjoint operator with compact resolvents, the Dirac operator.
4. We require $[D, x] \in B(H)$ for all $x \in \mathcal{A}$.

Our construction in superconformal field theory:

We construct a family $(\mathcal{A}(I), H, D)$ of spectral triples parametrized by intervals $I \subset S^1$ from a representation of the Ramond algebra. (Carpi-Hillier-K-Longo)

One of the Ramond relations gives $G_0^2 = L_0 - c/24$.
So G_0 should play the role of the Dirac operator, which is a "square root" of the Laplacian.
The representation space of the Ramond algebra is our Hilbert space H for the spectral triples (without a vacuum vector). The image of G_0 is now the Dirac operator D, common for all the spectral triples.

Then $\mathcal{A}(I) = \{x \in A(I) \mid [D, x] \in B(H)\}$ gives a net of spectral triples $\{\mathcal{A}(I), H, D\}$ parametrized by I.

$N = 2$ super Virasoro algebra (Ramond/N-S for $a = 0, 1/2$)

Generated by central element c, even elements L_n and J_n, and odd elements $G^{\pm}_{n\pm a}$, $n \in \mathbb{Z}$, with the following.

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \\
[J_m, J_n] &= \frac{c}{3} m \delta_{m+n,0} \\
[L_n, J_m] &= -mJ_{m+n}, \\
[L_n, G^{\pm}_{m\pm a}] &= \left(\frac{n}{2} - (m \pm a)\right) G^{\pm}_{m+n\pm a}, \\
[J_n, G^{\pm}_{m\pm a}] &= \pm G^{\pm}_{m+n\pm a}, \\
[G^{+}_{n+a}, G^{-}_{m-a}] &= 2L_{m+n} + (n - m + 2a)J_{n+m} + \\
&\quad \frac{c}{3}\left((n+a)^2 - \frac{1}{4}\right)\delta_{m+n,0}.
\end{aligned}
$$

An irreducible unitary representation maps c to a scalar in

$$\{3m/(m+2) | m = 1, 2, 3, \dots\} \cup [3, \infty).$$

We consider only the cases $c = 3m/(m+2)$.

We construct $N = 2$ superconformal nets from the vacuum representation of the Neveu-Schwarz algebra again with (super)stress energy tensors. This construction is natural and along the classical line in algebraic quantum field theory, but it is difficult to understand the representation theory.

The even part of the superconformal net is identified with the coset net for the inclusion $U(1)_{2m+4} \subset SU(2)_m \otimes U(1)_4$.

$\longrightarrow$ a good understanding of the representation theory

The irreducible representations of this local conformal net are labeled with triples (j, k, l) with $0 \leq j \leq m$, $0 \leq k < 2m + 4$, $0 \leq l < 4$ and $j - k + l \in 2\mathbb{Z}$ with the identification $(j, k, l) = (m - j, k + m + 2, l + 2)$,

The chiral ring and the spectral flow have been much studied in physics literatures in connection to the $N = 2$ superconformal field theory. In our context, the former is given by $\{(j, j, 0)\}$ and the latter is by $(0, 1, 1)$.

We classify all $N = 2$ superconformal nets with $c < 3$. More quantum fields give more operators, hence a larger von Neumann algebra. So look for possible extensions of $\{A(I)\}$, where $A(I)$ is generated by (super) stress energy tensor.

We have a general theory for such a classification based on α-induction and modular invariants, studied by Longo-Rehren, Xu, Böckenhauer-Evans-K. The classification of the modular invariants in this setting has been found by Gannon.

In similar classifications of local conformal nets and $N = 1$ superconformal nets, our classification lists consist of simple current extensions, the coset constructions, and the mirror extensions in the sense of Xu. (The last one is a new construction found with operator algebraic approach.)

In the $N = 2$ superconformal case, we have a mixture of the coset construction and the mirror extension, which is a new feature in this case.

Gepner model: Make a fifth tensor power of the $N = 2$ superconformal net with $c = 9/5$. Cyclic group actions give an example corresponding to a certain 3-dimensional Calabi-Yau manifold arising from a quintic in $\mathbb{CP}^4$.

This construction gives connection to the mirror symmetry. It appears as an isomorphism of two $N = 2$ super Virasoro algebras sending J_n to $-J_n$ and $G^{\pm}_m$ to $G^{\mp}_m$.

Some formal similarity to the Moonshine net: We have some basic net whose representation theory is well understood. Then make its finite tensor power, which should still be a basic example. Then finite group actions produce much more interesting examples.