Toeplitz CAR Flows

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Basics of E_0-semigroups

Definition

Let H be a separable infinite dimensional Hilbert space.
A semigroup $\alpha = \{\alpha_t\}_{t\geq 0}$ of unital endomorphisms of $B(H)$ is said to be an E_0-semigroup if the map $[0,\infty) \ni t \mapsto \alpha_t(A) \in B(H)$ is weakly continuous for every $A \in B(H)$.

Example (CAR flows)

Let $\{S_t\}$ be the shift semigroup of $L^2((0,\infty),\mathbb{C}^N)$.
The CAR flow of index N is an E_0-semigroup α acting on $B(\Gamma^a(L^2((0,\infty),\mathbb{C}^N)))$, which is determined by

$$\alpha_t(a(f)) = a(S_t f), \quad f \in L^2((0,\infty),\mathbb{C}^N).$$

where $\Gamma^a(L^2((0,\infty),\mathbb{C}^N))$ is the antisymmetric Fock space.

Two E_0-semigroups α acting on $B(H)$ and β acting on $B(K)$ are conjugate if there exists a unitary $U : H \to K$ satisfying $\mathrm{Ad}\, U \circ \alpha_t \circ \mathrm{Ad}\, U^* = \beta_t$.

An α-cocycle U is a weakly continuous map $[0, \infty) \ni t \mapsto U_t \in U(H)$ satisfying the cocycle relation $U_t \alpha_t(U_s) = U_{s+t}$.
The cocycle perturbation $\{\alpha_t^U = \mathrm{Ad}\, U_t \circ \alpha_t\}_{t \geq 0}$ of α by U is again an E_0-semigroup.

α and β are cocycle conjugate if a cocycle perturbation of α is conjugate to β.

Goal To classify E_0-semigroups up to cocycle conjugacy.

Definition

A unit of an E_0-semigroup α acting on $B(H)$ is a C_0-semigroup of isometries $V = \{V_t\}_{t\geq 0}$ on H satisfying $V_t A = \alpha_t(A) V_t$ for all $t \geq 0$.
An E_0-semigroup is said to be of

- type I if it has enough units.
- type II if it has a units, but they are not enough.
- type III (or unitless) if there is no unit.

An E_0-semigroup of either of type I or type II is called spatial.

Theorem (Arveson 89)

The CAR flows exhaust all type I E_0-semigroups up to cocycle conjugacy.

Type III Examples

Powers 87

First example of type III E_0-semigroup.
Ingredient: Quasi-free representation of CAR algebra.

Tsirelson 01

Uncountably many type III examples.
Ingredient: Off white noise.
c.f. CAR flows $\cong$ CCR flows. $\Gamma^s(L^2(0,\infty)) \cong L^2(\text{white noise})$.

I.-Srinivasan 08

Generalized CCR flows.
There exist uncountably many type III examples, which can not be distinguished from the CCR flow of index 1 by Tsirelson's invariant.

Toeplitz CAR flows

Let K be a complex Hilbert space.
The CAR algebra $\mathfrak{A}(K)$ is the C^*-algebra generated by $\{a(f)\}_{f\in K}$ such that $K \ni f \mapsto a(f) \in \mathfrak{A}(K)$ is linear and

$$a(f)a(g) + a(g)a(f) = 0$$

$$a(f)a(g)^* + a(g)^*a(f) = \langle f, g\rangle 1.$$

A (gauge invariant) quasi-free state $\omega_A \in S(\mathfrak{A}(K))$ associated with a positive contraction $A \in B(K)$ is determined by

$$\omega_A(a(f_n)\cdots a(f_1)a(g_1)^*\cdots a(g_m)^*) = \delta_{m,n}\det(\langle Af_i, g_j\rangle).$$

We denote by (π_A, H_A, Ω_A) the GNS triple of ω_A, and set $\mathcal{M}_A = \pi_A(\mathfrak{A}(K))''$.

- ω_A is a factor state.
- ω_A is a type I state if and only if $\mathrm{tr}(A - A^2) < \infty$.
- π_A and π_B are quasi-equivalent if and only if $A^{1/2} - B^{1/2}$ and $(1-A)^{1/2} - (1-B)^{1/2}$ are Hilbert-Schmidt operators.
- When $P \in B(H)$ is a projection, the restriction of π_A to $\mathfrak{A}(PK)$ is quasi-equivalent to π_{PAP}.

Let $K = L^2((0,\infty), \mathbb{C}^N)$, and let $\{S_t\}_{t\geq 0}$ be the shift semigroup. There exists a continuous semigroup $\{\rho_t\}_{t\geq 0} \subset \mathrm{End}(\mathfrak{A}(K))$ determined by $\rho_t(a(f)) = a(S_t f)$.

If $\omega_A \circ \rho_t = \omega_A$ (i.e $S_t^* A S_t = A$), ρ_t extends to $\mathcal{M}_A$.
If moreover $\mathrm{tr}(A - A^2) < \infty$, then $\{\rho_t\}_{t\geq 0}$ extends to an E_0-semigroup acting on the type I factor $\mathcal{M}_A$.

Let $\tilde{K} = L^2(\mathbb{R}, \mathbb{C}^N)$, and let $P_+ : \tilde{K} \to K$ be the projection K. For $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$, we denote by $C_\Phi \in B(\tilde{K})$ the Fourier multiplier multiplier

$$\widehat{C_\Phi f}(p) = \Phi(p)\widehat{f}(p), \quad f \in \tilde{K}.$$

The Toeplitz operator $T_\Phi \in B(K)$ with symbol Φ is defined by

$$T_\Phi f = P_+ C_\Phi f \quad f \in K.$$

The Hankel operator $H_\Phi \in B(K, K^\perp)$ with symbol Φ is defined by

$$H_\Phi f = (1 - P_+) C_\Phi f, \quad f \in K.$$

Lemma (Arveson)

Let A be a positive contraction of K.
Then A satisfies $S_t^* A S_t = A$ and $\mathrm{tr}(A - A^2) < \infty$ if and only if there exists a projection $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ such that $A = T_\Phi$ and H_Φ is Hilbert-Schmidt.

Proof.

Assume $S_t^* A S_t = A$ and $\mathrm{tr}(A - A^2) < \infty$.

$S_t^* A S_t = A$ implies that there exists a positive contraction $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ satisfying $A = T_\Phi$.

Since $A - A^2$ is compact and $A - A^2 = T_\Phi - T_\Phi^2 \geq T_{\Phi - \Phi^2}$, we get $\Phi = \Phi^2$.

Now $\mathrm{tr}(H_\Phi^* H_\Phi) = \mathrm{tr}(A - A^2) < \infty$ implies that H_Φ is Hilbert-Schmidt.

Definition

An admissible symbol Φ is a projection $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ with Hilbert-Schmidt Hankel operator H_Φ.
The Toeplitz CAR flow associated with an admissible symbol Φ, denoted by α^Φ, is the E_0-semigroup acting on $\mathcal{M}_{T_\Phi}$ extending $\{\rho_t\}_{t\geq 0}$.

If $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ is a constant projection, the Toeplitz CAR flow α^Φ is the CAR flow of index N.

Theorem (Powers 87)

Let $\Phi(p) = \frac{1}{2}\begin{pmatrix} 1 & e^{i\theta(p)} \\ e^{-i\theta(p)} & 1 \end{pmatrix}$ *with* $\theta(p) = \frac{1}{(1+p^2)^{1/5}}$.
Then Φ *is admissible and* α^Φ *is of type III.*

For $\varphi \in L^\infty(\mathbb{T})$, it is well-known that H_φ and $H_{\overline{\varphi}}$ are Hilbert-Schmidt iff φ is in the Sobolev space $W_2^{1/2}(\mathbb{T})$, that is,

$$\int_{\mathbb{T}^2} \frac{|\varphi(e^{is}) - \varphi(e^{it})|^2}{|e^{is} - e^{it}|^2} dsdt < \infty.$$

Lemma

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ be a projection.

- *Φ is admissible iff*

$$\int_{\mathbb{R}^2} \frac{\|\Phi(p) - \Phi(q)\|_2^2}{|p-q|^2} dpdq < \infty.$$

- *If Φ is admissible, $\int_{\mathbb{R}} \|\Phi(2p) - \Phi(p)\|_2^2 \frac{dp}{|p|} < \infty$.*
- *If Φ is an even differential function satisfying $\int_0^\infty \|\Phi'(p)\|_2^2 p dp < \infty$, then Φ is admissible.*

Corollary

Let θ *be a even differential real function on* $\mathbb{R}$ *satisfying* $\int_0^\infty |\theta'(p)|^2 p dp < \infty$*.*
Then $\Phi(p) = \frac{1}{2}\begin{pmatrix} 1 & e^{i\theta(p)} \\ e^{-i\theta(p)} & 1 \end{pmatrix}$ *is an admissible symbol.*

$\theta(p) = \log\log(r + p^2)$ with $r > 1$ satisfies the above condition.

$\theta(p) = \frac{1}{(1+p^2)^\lambda}$ with $\lambda > 0$ satisfies the above condition.
We denote by α^λ the corresponding Toeplitz CAR flows.
$\alpha^{1/5}$ is Powers's example of type III E_0-semigroup.

Questions

What is the type of α^λ ?
If $\lambda_1 \neq \lambda_2$ and α^{λ_1} and α^{λ_2} are of type III, are they different?

Main result

Theorem (I.-Srinivasan 2010)

Let $\lambda > 0$, and let α^λ be the Toeplitz CAR flow with symbol
$\Phi(p) = \frac{1}{2}\begin{pmatrix} 1 & e^{i\theta(p)} \\ e^{-i\theta(p)} & 1 \end{pmatrix}$, $\theta(p) = \frac{1}{(1+p^2)^\lambda}$.

Then

- *If $\lambda > 1/4$, α^λ is cocycle conjugate to the CAR flows of index 2.*
- *If $\lambda \leq 1/4$, α^λ is of type III.*
- *If $0 < \lambda_1 < \lambda_2 \leq 1/4$, α^{λ_1} and α^{λ_2} are not cocycle conjugate.*

Type I Criterion

Theorem (Powers 87, Arveson 2003)

If admissible $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ *has limit at* ∞*, and* α^Φ *is spatial,*

$$\int_{\mathbb{R}} \|\Phi(p) - \Phi(\infty)\|_2^2 dp < \infty.$$

Lemma (I.-Srinivasan 2010)

If two admissible symbols $\Phi, \Psi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ *satisfy*

$$\int_{\mathbb{R}} \|\Phi(p) - \Psi(p)\|_2^2 dp < \infty,$$

α^Φ *and* α^Ψ *are cocycle conjugate.*

Theorem (I.-Srinivasan 2010)

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ *be an admissible symbol.*
The following 3 conditions are equivalent:

- α^Φ *is cocycle conjugate to the CAR flow of index* N*.*
- α^Φ *is spatial.*
- *There exists a constant projection* $Q \in M_N(\mathbb{C})$ *such that*

$$\int_{\mathbb{R}} \|\Phi(p) - Q\|_2^2 dp < \infty.$$

Invariant

A type I factorization of $B(H)$ is a family $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ of type I subfactors of $B(H)$ such that $\mathcal{M}_\lambda$ commutes with $\mathcal{M}_\mu$ for $\lambda \neq \mu$, and $B(H) = \bigvee_{\lambda\in\Lambda} \mathcal{M}_\lambda$.

A type I factorization $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ of $B(H)$ is said to be a complete atomic Boolean algebra of type I factors (CABATIF) if $\bigvee_{\lambda\in\Gamma} \mathcal{M}_\lambda$ is a type I factor for every $\Gamma \subset \Lambda$.

Theorem (Araki-Woods 66)

For a type I factorization $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ of $B(H)$, the following 3 conditions are equivalent:

- $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ *is a CABATIF.*
- $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ *has a factorizable vector.*
- $\{\mathcal{M}_\lambda\}_{\lambda\in\Lambda}$ *is a tensor product factorization.*

Let α be an E_0-semigroup acting on $B(H)$.

For $0 \leq s < t$, we set $\mathcal{A}^\alpha(s,t) = \alpha_s(B(H)) \cap \alpha_t(B(H))'$, which is a type I factor.

Let $\{a_n\}_{n=0}^\infty$ be a strictly increasing sequence of numbers with $a_0 = 0$ converging to $a < \infty$.

$\{\mathcal{A}^\alpha(a_n, a_{n+1})\}_{n=0}^\infty$ is a type I factorization of $\mathcal{A}^\alpha(0,a)$.

For a fixed $\{a_n\}_{n=0}^\infty$, whether $\{\mathcal{A}^\alpha(a_n, a_{n+1})\}_{n=0}^\infty$ is a CABATIF or not is a cocycle conjugacy invariant for α.

Theorem (I.-Srinivasan 2010)

Let $\Phi \in L^\infty(\mathbb{R}) \otimes M_N(\mathbb{C})$ *be an admissible symbol satisfying* $\Phi(p) = \Phi(-p)$, *and let* $0 < \mu < 1$.

We set $a_0 = 0$, $a_n = \sum_{k=1}^{n} \frac{1}{k^{1/(1-\mu)}}$ *for* $n \in \mathbb{N}$, *and* $a = \lim_{n\to\infty} a_n$.

Then the following two conditions are equivalent

(1) $\{\mathcal{A}^{\alpha^\Phi}(a_n, a_{n+1})\}_{n=0}^\infty$ *is a CABATIF.*

(2) $\int_{\mathbb{R}^2} \frac{\|\Phi(p)-\Phi(q)\|_2^2}{|p-q|^{1+\mu}} dpdq < \infty$.

Moreover,

- *If* $\{\mathcal{A}^{\alpha^\Phi}(a_n, a_{n+1})\}_{n=0}^\infty$ *is a CABATIF,* $\int_0^\infty \|\Phi(2p) - \Phi(p)\|_2^2 \frac{dp}{p^\mu} < \infty$.
- *If* Φ *is differential and* $\int_0^\infty \|\Phi'(p)\|_2^2 p^{2-\mu} dp < \infty$, $\{\mathcal{A}^{\alpha^\Phi}(a_n, a_{n+1})\}_{n=0}^\infty$ *is a CABATIF.*