Wenzl's theorem

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Our goal in this lecture is to indicate a proof of the following result of Wenzl, which was inspired by the result of Jones on restriction of index values:

Theorem 1: (Wenzl) If there exists a sequence $\{e_n : n = 1, 2, \cdots\}$ of orthogonal projections on Hilbert space, which satisfy the relations defining $TL(\tau)$, then

$$\tau \in (0, \frac{1}{4}] \cup \{\frac{1}{4} sec^2(\frac{\pi}{n}) : n = 3, 4, 5, \cdots\}$$

But we first need a digression into traces, conditional expectations, and a variant of Tchebyshev polynomials of the second kind.

Definition: A linear functional 'tr' on an algebra A is said to be

- a *trace* if tr (xy) = tr (yx) forall $x, y \in A$;
- *normalised* if A is unital and tr(1) = 1;
- *positive* if A is a *-algebra and tr $(x^*x) \geq 0 \forall x \in A$;
- *faithful and positive* if A is a *-algebra and tr $(x^*x) > 0 \; \forall \; 0 \neq x \in A$.

For example, $M_n(\mathbb{C})$ admits a unique normalised trace (tr$(x) = \frac{1}{n}\sum_{i=1}^{n} x_{ii}$) which is automatically faithful and positive.

Proposition FDC*: The following conditions on a finite-dimensional unital *-algebra A are equivalent:

1. There exists a unital *-isomorphism from $\pi : A \to M_n(\mathbb{C})$ for some n.

2. There exists a faithful positive normalised trace on A.

Proof: (1) $\Rightarrow$ (2): Set $\mathrm{tr}_A = \mathrm{tr}_{M_n(\mathbb{C})} \circ \pi$

(2) $\Rightarrow$ (1): Set $H = \{\widehat{x} : x \in A\}$, define

$$\langle \widehat{x}, \widehat{y} \rangle = \mathrm{tr}(y^* x),$$

and note that H becomes an inner product space.

Consider the map $\pi : A \to End_{\mathbb{C}}(H)$ defined by

$$\pi(x)\hat{y} = \widehat{xy}$$

Observe that π is an algebra homomorphism, such that

$$\langle \pi(x)\hat{y}, \hat{z} \rangle = \mathrm{tr}(z^* x y) = \mathrm{tr}((x^* z)^* y) = \langle \hat{y}, \pi(x^*)\hat{z} \rangle$$

i.e., $\pi(x)^* = \pi(x^*)$.

The fact that A has a unit implies that π is faithful (since $\pi(x) = 0 \Rightarrow \mathrm{tr}(x^* x) = \|\hat{x}\|^2 = \|\pi(x)\hat{1}\|^2 = 0 \Rightarrow x = 0$. Finally, setting $n = dim(H) = dim(A)$, and realising linear operators on H as matrices with respect t some orthonormal basis of H, we may view π as a faithful *-homomorphism into $M_n(\mathbb{C})$. □

Note: A *-algebra A as in the above Proposition is nothing but a finite-dimensional C^*-algebra. Such an A may admit several faithful positive normalised traces in general.

Suppose $A_0 \subset A$ is a unital inclusion of finite-dimensional C^*-algebras, and suppose 'tr' is a faithful positive normalised trace on A. Let $H = \{\hat{a} : a \in A\}$ be the finite-dimensional Hilbert space as above, and let us simply identify $x \in A$ with $\pi(x) \in End_{\mathbb{C}}(H)$ - so that $x\hat{y} = \widehat{xy}$. (The artificial looking 'hat's were introduced in order to distinguish between x, the operator on H and $\hat{x}$, the vector in H.) Let $H_0 = \{\widehat{a_0} : a_0 \in A_0\}$ and let e_{A_0} denote the orthogonal projection of H onto the subspace H_0. Since faithfulness of 'tr' translates into injectivity of the map $A \ni a \mapsto \hat{a} \in H$, we see that there exists a uniquely defined $\mathbb{C}$-linear map $E_{A_0} : A \to A_0$, usually called *the* 'tr'-*preserving conditional expectation of A onto A_0*, such that $e_{A_0}(\hat{a}) = \widehat{E_{A_0}a}$. The following facts may be verified to hold, for all $a, b \in A, a_0, b_0 \in A_0$:

$$\begin{aligned} E_{A_0}(a_0 b b_0) &= a_0 E_{A_0}(b) b_0 \\ E_{A_0}(a_0) &= a_0 \\ \mathrm{tr}|_{A_0} \circ E_{A_0} &= \mathrm{tr} \\ e_{A_0} a e_{A_0} &= (E_{A_0} a) e_{A_0} \end{aligned}$$

There is a natural *-structure on $TL_n(\beta^{-2}) = D_n(\beta)$ with the adjoint T^* of a Kauffman diagram T being defined as the diagram obtained by reflecting T about a horizontal lilne in the middle of the bounding box. Thus, E_i is self-adjoint for each i.

Also, there is a natural inclusion (= unital *-algebra monomorphism) of TL_n into TL_{n+1} which maps e_i to e_i for $1 \leq i < n$. At the level of diagrams, it identifies a $T \in \mathcal{K}_n$ with the element of $\mathcal{K}_{n+1}$ btained by adding on a vertical strand to the right end of T.

Although the TL_n's are not quite C^*-algebras in general, they nevertheles come equipped with a consistent family of traces {tr} and consistent conditional expectations $\epsilon_n : D_{n+1}(\beta) \to D_n(\beta)$ as follows:

If a is an $(n+1, n+1)$ diagram, then $\tilde{\epsilon_n}(a)$ is obtained by just closing up the last strand. Hence if $a \in D_n(\beta)$ then $\tilde{\epsilon_n}(a) = \beta a$. Define $\epsilon_n(a) = \frac{1}{\beta}\tilde{\epsilon_n}(a)$ for $a \in D_n(\beta)$. Then ϵ_n is a conditional expectation.

Let $tr_n : D_n(\beta) \to \mathbb{C}$ be defined by $tr_n(a) = (\epsilon_1\epsilon_2 \cdots \epsilon_{n-1})(a)$. Note that $tr_n(a) = tr_{n+1}(a)$ if $a \in D_n(\beta)$. Hence we can and will denote tr_n by tr. If a is a diagram, let $c(a)$ be the number of loops one gets when one closes all the strands. Then $tr(a) = \beta^{c(a)-n}$

$tr : D_n(\beta) \to \mathbb{C}$ is a unital trace and satisfies the following properties:

1. $tr(x) = tr(\epsilon_n(x)) \ \forall \ x \in D_{n+1}(\beta)$.

2. $e_n x e_n = \epsilon_{n-1}(x) e_n \ \forall \ x \in D_n(\beta)$.

3. $tr(e_i) = \tau$ where $\tau = \frac{1}{\beta^2}$.

The following variants of *Tchebyshev polynomials of the second kind* are important for us:

$$
\begin{aligned}
P_0(x) &= P_1(x) = 1 && (1) \\
P_{n+1}(x) &= P_n(x) - xP_{n-1}(x) && (2)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= 1 \\
P_2(x) &= 1 - x \\
P_3(x) &= 1 - 2x \\
P_4(x) &= 1 - 3x + x^2 \\
P_4(x) &= 1 - 4x + 3x^2 \\
P_5(x) &= 1 - 5x + 6x^2 - x^3 \\
P_6(x) &= 1 - 6x + 10x^2 - 4x^3 \\
- - - &\quad - - -
\end{aligned}
$$

Lemma P_n:

If we set

$$\sigma = \frac{1+\sqrt{1-4x}}{2}, \overline{\sigma} = \frac{1-\sqrt{1-4x}}{2}$$

we have

(1) $P_n(x) = \frac{\sigma^{n+1}-\overline{\sigma}^{n+1}}{\sigma-\overline{\sigma}}$

(2) $P_n(\frac{1}{4}sec^2\theta) = \frac{sin(n+1)\theta}{2^n cos^n\theta sin\theta}$

(3) The polynomial P_n is of degree $m = [\frac{n}{2}]$. It's leading coefficient is $(-1)^m$ if $n = 2m$ and $(-1)^m(m+1)$ if $n = 2m+1$.

(4) The polynomial P_n has distinct zeros given by $\{\frac{1}{4}sec^2(\frac{\pi j}{n+1}) : 1 \leq j \leq m\}$

(5) If $n \geq 2$ and if $\frac{1}{4}sec^2(\frac{\pi}{n+2}) < \lambda < \frac{1}{4}sec^2(\frac{\pi}{n+1})$, then $P_i(\lambda) > 0$ for $1 \leq i \leq n$ and $P_{n+1}(\lambda) < 0$.

Proof: (1) Note that σ and $\overline{\sigma}$ are the roots of the equation $p^2 - p + x = 0$, so the general solution of the recurrence relation defining the P_k's is seen to be $P_n = A\sigma^{n+1} + B\overline{\sigma}^{n+1}$; the 'boundary conditions' demand that $A + B = 0$ (for $n = -1$) and $A\sigma + B\overline{\sigma} = 1$ (for $n = 0$); this yields (1).

(2) Setting $x = \frac{1}{4}sec^2\theta$, we find that $\sigma = re^{i\theta}, \overline{\sigma} = re^{-i\theta}$ where $r = \frac{1}{2cos\theta}$, and hence $\sigma^{n+1} - \overline{\sigma}^{n+1} = 2ir^{n+1}sin(n+1)\theta, \sigma - \overline{\sigma} = 2irsin\theta$, thereby establishing (2).

(3) This is shown fairly easily by induction, using the recurrence relation satisfied by the P_n's.

(4) It follows from (2) that the numbers $\frac{1}{4}sec^2(\frac{\pi j}{n+1})$ yield m distinct zeros of P_n. Since P_n has degree m, this assertion is clear.

(5) It is seen from (2) that $lim_{x\to-\infty}P_n(x) = +\infty$ for all n; in particular, P_n is positive to the left of its first zero, and since the function $x \mapsto sec^2(x)$ is an increasing function in $(0, \frac{\pi}{2}$, it is seen that for all $k \leq n$ and $j \leq [frack2]$, we have

$$\begin{aligned}
\lambda &< \frac{1}{4}sec^2(\frac{\pi}{n+1}) \\
&< \frac{1}{4}sec^2(\frac{\pi}{k+1}) \\
&< \frac{1}{4}sec^2(\frac{j\pi}{k+1})
\end{aligned}$$

and consequently λ lies to the left of the first zero of P_k, whence $P_k(\lambda) > 0$.

On the other hand, the inequalities

$$\frac{1}{4}sec^2(\frac{\pi}{n+2}) < \lambda < \frac{1}{4}sec^2(\frac{\pi}{n+1}) < \frac{1}{4}sec^2(\frac{2\pi}{n+2})$$

show that λ lies between the first two zeros, and we may conclude that indeed $P_{n+1}(\lambda) < 0$.

Let $TL(\tau) = \bigcup_n T_n(\tau)$. Then $TL(\tau)$ is a $\star$ algebra generated by $1, e_1, e_2, \ldots$. When $\tau > 0$, e_i's are self adjoint.

Lemma JW:(Wenzl) Let τ be a nonzero complex number such that $P_k(\tau) \neq 0$ for $k = 1, 2, \cdots, n$. Define (the so-called **Jones-Wenzl idempotents**) f_k in $TL(\tau)$ recursively as follows:

$$\begin{aligned} f_0 = f_1 &= 1 \\ f_{k+1} &= f_k - \frac{P_{k-1}(\tau)}{P_k(\tau)} f_k e_k f_k, \ 1 \le k \le n. \end{aligned}$$

Then, for $1 \le k \le n+1$, we have:

(1) $f_k \in T_k(\tau)$.

(2) If $k \geq 2$, then $1 - f_k$ is in the algebra generated by $\{e_1, \cdots, e_{k-1}\}$

(3) $(e_k f_k)^2 = \frac{P_k(\tau)}{P_{k-1}(\tau)} e_k f_k$, $(f_k e_k)^2 = \frac{P_k(\tau)}{P_{k-1}(\tau)} f_k e_k$,

(4) f_k is an idempotent.

(5) $f_k e_i = 0$, $e_i f_k = 0$ if $i \leq k-1$.

(6) $tr(f_k) = P_k(\tau)$.

When $\tau > 0$, f_k is selfadjoint.

Proof: The proof is by induction on k. Assertions $1-6$ are clearly true for $k \leq 2$. Now assume that $1-6$ are valid for $1 \leq k \leq l$ where $l \geq 2$. We will show the result is true for $k = l+1$.

Since f_l is in $T_l(\tau)$, it follows by definition that f_{l+1} is in the algebra generated by $1, e_1, e_2, \cdots, e_l$. Hence $f_{l+1} \in T_{l+1}(\tau)$. Since $1 - f_l$ is in the algebra genrated by $e_1, e_2, \cdots, e_{l-1}$, by definition, it follows that $1 - f_{l+1}$ is in the algebra generated by $e_1, e_2, \cdots, e_l$.

Now note that $f_{l+1}f_l = f_{l+1}$ and $f_l f_{l+1} = f_{l+1}$ since f_l is an idempotent. Since $f_l \in T_l(\tau)$, e_{l+1} commutes with f_l. Thus,

$$\begin{aligned} e_{l+1}f_{l+1}e_{l+1} &= e_{l+1}f_l - \frac{P_{l-1}(\tau)}{P_l(\tau)} f_l e_{l+1} e_l e_{l+1} f_l \\ &= \frac{P_{l+1}(\tau)}{P_l(\tau)} e_{l+1} f_l \end{aligned}$$

Hence $(e_{l+1}f_{l+1})^2 = \frac{P_{l+1}(\tau)}{P_l(\tau)} e_{l+1}f_{l+1}$.

The proof that $(f_{l+1}e_{l+1})^2 = \frac{P_{l+1}(\tau)}{P_l(\tau)} f_{l+1}e_{l+1}$ is similar.

Next

$$\begin{aligned} & f_{l+1}^2 \\ &= f_l^2 - 2\frac{P_{l-1}(\tau)}{P_l(\tau)} f_l e_l f_l + \left(\frac{P_{l-1}(\tau)}{P_l(\tau)}\right)^2 f_l e_l f_l e_l f_l \\ &= f_l^2 - 2\frac{P_{l-1}(\tau)}{P_l(\tau)} f_l e_l f_l + \left(\frac{P_{l-1}(\tau)}{P_l(\tau)}\right)^2 \frac{P_l(\tau)}{P_{l-1}(\tau)} f_l e_l f_l \\ &= f_l - \frac{P_{l-1}(\tau)}{P_l(\tau)} f_l e_l f_l = f_{l+1} \end{aligned}$$

Hence f_{l+1} is an idempotent.

Since $f_{l+1}e_i = f_{l+1}f_l e_i$, it follows that $f_{l+1}e_i = 0$ if $i \leq l-1$. Now $f_{l+1}e_l = f_l e_l - \frac{P_{l-1}(\tau)}{P_l(\tau)}(f_l e_l)^2$. But $(f_l e_l)^2 = \frac{P_l(\tau)}{P_{l-1}(\tau)} f_l e_l$, and so $f_{l+1}e_l = 0$. Hence $f_{l+1}e_i = 0$ for $i \leq l$. Similarly $e_i f_{l+1} = 0$.

Next,

$$
\begin{aligned}
tr(f_{l+1}) &= tr(f_l) - \frac{P_{l-1}(\tau)}{P_l(\tau)} tr(f_l e_l f_l) \\
&= tr(f_l) - \frac{P_{l-1}(\tau)}{P_l(\tau)} tr(\epsilon_l(f_l e_l f_l)) \\
&= tr(f_l) - \frac{P_{l-1}(\tau)}{P_l(\tau)} tr(f_l \epsilon_l(e_l) f_l) \\
&= tr(f_l) - \frac{P_{l-1}(\tau)}{P_l(\tau)} tr(\tau f_l) \\
&= P_l(\tau) - \tau P_{l-1}(\tau) = P_{l+1}(\tau)
\end{aligned}
$$

If $\tau > 0$ then $P_k(\tau)$ is real. Hence by induction it follows that $f_k' s$ are selfadjoint. $\square$

We shall next prove the following lemma, before proceeding to prove Wenzl's theorem.

Lemma 1: Let τ be such that $\frac{1}{4}sec^2(\frac{\pi}{n+2}) < \tau < \frac{1}{4}sec^2(\frac{\pi}{n+1})$ for some $n \in \mathbb{N}$, with $n \geq 2$. Suppose $\pi : TL(\tau) \to B(H)$ be a $\star$ homomorphism, where H is a Hilbert space. Let e_i^T denote the idempotents in $TL(\tau)$. Then the Jones-Wenzl idempotents f_k^T's are defined for $k = 1, 2, \cdots n+2$. Suppose $f_k = \pi(f_k^T)$ for $k \leq n+2$. Then

(1) $1 - f_k = e_1 \vee e_2 \vee \cdots e_{k-1}$ for $k \leq n+2$.

(2) $e_{n+1} f_{n+1} = 0$.

(3) e_{n+1} is orthogonal to f_n.

Proof: Note that $P_k(\tau) > 0$ for $k = 1, 2, \cdots n$ and $P_{n+1}(\tau) < 0$. Hence the Jones-Wenzl idempotents are defined for $k = 1, 2, \cdots n + 2$.

By Lemma JW, it follows that $f_k e_i = 0$ for $i \leq k - 1$. Hence we have $e_1 \vee e_2 \vee \cdots \vee e_{k-1} \leq 1 - f_k$. Since $1 - f_k$ is in the algebra generated by $e_1, e_2, \cdots, e_{k-1}$, it follows that $1 - f_k \leq e_1 \vee e_2 \vee \cdots e_{k-1}$. This proves (1).

Observe that $e_{n+1} f_{n+1} e_{n+1} = \frac{P_{n+1}(\tau)}{P_n(\tau)} e_{n+1} f_n$. But $e_{n+1} f_{n+1} e_{n+1}$ is positive and $e_{n+1} f_n$ is a projection. Since $P_{n+1}(\tau) < 0$, it follows that $e_{n+1} f_n = 0$ and $(f_{n+1} e_{n+1})^{\star} f_{n+1} e_{n+1} = 0$. Hence $f_{n+1} e_{n+1} = 0$ and e_{n+1} is orthogonal to f_n. By taking adjoints, we get $e_{n+1} f_{n+1} = 0$. This proves (2) and (3). $\square$

Proposition (orth): Let H be a Hilbert space. Suppose $e_1, e_2, \cdots$ is a sequence of non-zero projections in $B(H)$ satisfying the following relation :

$$\begin{aligned} e_i^2 &= e_i = e_i^\star \\ e_ie_j &= e_je_i = 0 \qquad \text{if} \quad |i-j| \geq 2 \\ e_ie_je_i &= \tau e_i \qquad\qquad \text{if} \quad |i-j| = 1 \end{aligned}$$

Then $\tau \in (0, \frac{1}{4}] \cup \{\frac{1}{4}sec^2(\frac{\pi}{n+1}) : \; n \geq 2\}$.

Proof: There exists a nontrivial $C^\star$ representation of $TL(\tau)$ say π which is unital and for which $\pi(e_i^T) = e_i$ where e_i^T denote the idempotents in $TL(\tau)$. By taking norms on the third relation, it follows that $\tau \leq 1$. Suppose that τ is not in the set given in the proposition. Then there exists $n \geq 2$ such that $\frac{1}{4}sec^2(\frac{\pi}{n+2}) < \tau < \frac{1}{4}sec^2(\frac{\pi}{n+1})$. Then $P_k(\tau) > 0$ for $k = 1, 2, \cdots n$ but $P_{n+1}(\tau) < 0$. Hence, the Jones Wenzl idempotents f_k^T's are defined for $k = 1, 2, \cdots n+2$. Let $f_k = \pi(f_k^T)$ for $k \leq n+2$.

By Lemma 1, it follows that e_{n+1} is orthogonal to f_n. But e_{n+1} is orthogonal to $e_1 \vee e_2 \vee \cdots e_{n-1}$ which latter projection is, by Lemma 1, nothing but $1-f_n$. Hence $e_{n+1} = e_{n+1}f_n + e_{n+1}(1-f_n) = 0$ which is a contradiction. This completes the proof. $\square$

Proof of Wenzl's theorem:

Suppose that τ is not in the set described above. Then there exists $n \geq 2$ such that $\frac{1}{4}sec^2(\frac{\pi}{n+2}) < \tau < \frac{1}{4}sec^2(\frac{\pi}{n+1})$. From lemma **??**, it follows that $e_{n+1}f_{n+1} = 0$. Also $e_i f_{n+1} = 0$ for $i \leq n$. Hence $f_{n+1} \leq 1 - e_1 \vee e_2 \vee \cdots \vee e_{n+1} = f_{n+2}$. But $f_{n+2} \leq f_{n+1}$. Hence $f_{n+1} = f_{n+2}$. Let k be the least element in $\{2, 3, \cdots, n\}$ for which $f_{k+1} = f_{k+2}$. Let $g_i = e_{k+i}f_{k-1}$ for $i \geq 0$. We will derive a contradiction by showing that $g_i's$ satisfy the hypothesis of Proposition (orth).

Since e_{k+i} commutes with f_{k-1} for $i \geq 0$, it follows that g_i's are projections. For the same reason, $g_i's$ satisfy the third relation of Proposition (orth). First, we show that $g_0 \neq 0$. By the choice of k, $f_k \neq f_{k+1}$. Hence $f_k e_k f_k \neq 0$. Since $f_k \leq f_{k-1}$, it follows that $f_{k-1} e_k = g_0 \neq 0$.

Now we show that $g_i g_j = 0$ if $|i-j| \geq 2$. We begin by showing $g_0 g_2 = 0$. Observe that since $f_{k+1} = f_{k+2}$, we have

$$e_{k+1} f_k = e_{k+1}(f_k - f_{k+1}) e_{k+1} = e_{k+1}\left(\frac{P_{k-1}(\tau)}{P_k(\tau)} f_k e_k f_k\right) e$$

Since $P_{k+1}(\tau) \neq 0$, it follows that $e_{k+1} f_k = 0$. By premultiplying and postmultiplying by e_{k+2}, we see that $e_{k+2} f_k = 0$. Hence we have,

$$
\begin{aligned}
g_0 g_2 &= e_k e_{k+2} f_{k-1} \\
&= e_k e_{k+2}(f_{k-1} - f_k) e_{k+2} e_k \\
&= e_{k+2} e_k (f_{k-1} - f_k) e_k e_{k+2} \\
&= e_{k+2} e_k \left(\frac{P_{k-2}(\tau)}{P_{k-1}(\tau)} f_{k-1} e_{k-1} f_{k-1}\right) e_k e_{k+2} \\
&= \tau \frac{P_{k-2}(\tau)}{P_{k-1}(\tau)} g_0 g_2
\end{aligned}
$$

Since $P_k(\tau) \neq 0$, it follows that $g_0 g_2 = 0$. Let $i \geq 2$. Let us consider the partial isometry $w = (\frac{1}{\tau})^{i-1} e_{k+i} e_{k+i-1} \cdots e_{k+2}$. Since w commutes with e_k and f_{k-1}, $w e_k f_{k-1}$ is a partial isometry. Note that $(w e_k f_{k-1})^\star w e_k f_{k-1} = g_0 g_2 = 0$. Thus, $g_i g_0 = w e_k f_{k-1} (w e_k f_{k-1})^\star = 0$. Hence $g_i g_0 = 0$ if $i \geq 2$. Let i, j be such that $j \geq i+2$. Now let $u = (\frac{1}{\tau})^{i+1} e_{k+i} e_{k+i-1} \cdots e_k$. Then u is a partial isometry which commutes with f_{k-1} and e_{k+j}.

Let $v = ue_{k+j}f_{k-1}$. Then v is a partial isometry such that $v^\star v = g_0 g_j$ and $vv^\star = g_i g_j$. Since $v^\star v = 0$, it follows that $vv^\star = 0$. Thus $g_i g_j = 0$. Therefore g_i 's satisfy the assumptions of Proposition (orth). Hence we have a contradiction. This completes the proof. $\square$