Home-work 8

on lecture dated 07/11/09

1. (a) For arbitrary $c \in \mathbb{R}$, define

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

compute the matrix product CA, where $A = ((a^i_j)) \in M_{4\times 6}(\mathbb{R})$ is arbitrary.

(b) More generally, for arbitrary positive integers $1 \leq k, l \leq m$, with $k \neq l$, and $c \in \mathbb{R}$, define $R^{(m)}_{k,l}(c)$ to be the $m \times m$ matrix with (ij) entry given by $\delta_{i,j} + c\delta_{k,i}\delta_{l,j}$, where the **Kronecker delta** is defined by

$$\delta_{p,q} = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}.$$

(Thus $R^{(4)}_{2,4}(c)$ is what was called C in (a) above.) If n is any positive integer, compute the matrix product $R^{(m)}_{k,l}(c)A$, where $A = ((a^i_j)) \in M_{m\times n}(\mathbb{R})$ is arbitrary.

2. (a) Define

$$F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

compute the matrix product FA, where $A = ((a^i_j)) \in M_{4\times 6}(\mathbb{R})$ is arbitrary.

(b) More generally, for arbitrary positive integers $1 \leq k, l \leq m$, with $k \neq l$, define $F^{(m)}_{kl}$ to be the $m \times m$ matrix with (ij) entry given by

$$f^i_j = \begin{cases} 1 & \text{if } i = j \neq k \\ 1 & \text{if } i = k \text{ and } j = l \\ 0 & \text{otherwise} \end{cases}.$$

(Thus $F^{(4)}_{13}$ is what was called F in (a) above.) If n is any positive integer, compute the matrix product $F^{(m)}_{kl}A$, where $A = ((a^i_j)) \in M_{m\times n}(\mathbb{R})$ is arbitrary.

3. The goal of this problem is to determine the matrix with respect to the standard basis of the operator $T \in L(\mathbb{R}^3)$ of reflection in the plane $\Pi = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$. (Thus $Tv = v$ if $v \in \Pi$ and Tx is the mirror-image of x obtained by reflection in Π.)

(a) Show that

$$\mathcal{B} = \{v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, v_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, v_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\}$$

is a basis for $\mathbb{R}^3$ which is 'well-adapted' to T in the sense that $\{v_1, v_2\}$ is a basis for Π, while $v_3 \perp \Pi$. Deduce that

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(b) Let

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Verify that U is an invertible matrix whose inverse is its *transpose*; i.e.,

$$U^{-1} = U' = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

(c) Observe that $U = [S]_{\mathcal{E}}$ where

$$\mathcal{E} = \{e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$$

is the standard basis, and $S \in L(\mathbb{R}^3)$ is the unique (and necessarily invertible) operator satisfying $Se_i = v_1, \ \forall 1 \leq i \leq 3$.

(d) Deduce that $[T]_{\mathcal{E}} = U[T]_{\mathcal{B}}U'$, and hence compute $[T]_{\mathcal{E}}$.