Home-work 7

on lecture dated 24/10/09

In the following exercises, the symbols V_1, V_2 denote vector spaces, $L(V_1, V_2)$ denotes the set of all linear transformations from V_1 to V_2, T denotes an arbitrary element of $L(V_1, V_2)$, and 0 denotes the 'zero vector' in any vector space.

1. Show that $T0 = 0$.

2. Prove that T is one-to-one if and only if $\{x \in V_1 : Tx = 0\} = \{0\}$.

3. Verify that the transformation of $\mathbb{R}^2$ given by anti-clockwise rotation by an angle θ about the origin is represented by the matrix given by

$$R_\theta = \begin{bmatrix} cos\ \theta & -sin\ \theta \\ sin\ \theta & cos\ \theta \end{bmatrix}$$

4. Since $R_{\theta+\phi} = R_\theta R_\phi$, deduce the *sum formulae* for trigonometric functions:

$$\begin{aligned} cos(\theta + \phi) &= cos\ \theta\ cos\ \phi\ -\ sin\ \theta\ sin\ \phi \\ sin(\theta + \phi) &= sin\ \theta\ cos\ \phi\ +\ cos\ \theta\ sin\ \phi \end{aligned}$$

5. If $M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ denotes the matrix corresponding to reflection in the 'x-axis', then $MR_\theta = R_\theta M$ if and only if $\theta = 0$ or $\theta = \pi$. In particular, matrix multiplication is *not* commutative.

6. Prove that the equation

$$T(a + ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

defines a vector space isomorphism of $\mathbb{C}$ onto $T(\mathbb{C}) \subset M_2(\mathbb{R})$ which also preserves products.