**Home-work 5**

*on lecture dated 03/10/09*

In the following, assume $V$ is an $n$-dimensional space, and that $W_1, W_2$ denote subspaces of $V$

1. Show that $W_1 \cap W_2$ contains a non-zero vector if $dim(W_1) + dim(W_2) > n$. Deduce that any two planes through the origin in $\mathbb{R}^3$ meet in a straight line if they are not the same.

2. (a) Give examples of a pair of subspaces such that $dim(W_1) + dim(W_2) = n$ where $W_1 \cap W_2 = \{0\}$

   (b) Show that two subspaces can satisfy the criteria of the previous formula if and only if every vector $v \in V$ is *uniquely* expressible in the form $v = w_1 + w_2$ with $w_i \in W_i, i = 1, 2$. (When this happens, we say $V = W_1 \oplus W_2$ is the 'internal direct sum' of the subspaces $W_1$ and $W_2$.)

3. Given vector spaces $U$ and $V$, write

$$U \oplus V = \{(u, v) : u \in U, v \in V\}$$

   (a) Verify that $U \oplus V$ is a vector space with respect to 'coordinate-wise operations' defined by $\alpha(u_1, v_1) + (u_2, v_2) = (\alpha u_1 + u_2, \alpha v_1 + v_2)$ for all $(u_1, v_1), (u_2, v_2) \in U \oplus V$.

   (b) Define $U_1 = \{(u, 0) \in U \oplus V : u \in U\}$ and $V_1 = \{(0, v) \in U \oplus : v \in V\}$. Verify that $U \oplus V$ is the internal direct sum of $U_1$ and $V_1$.

   (c) Show that the external direct sum $(U \oplus V)$ is 'isomorphic' to the internal direct sum of $U_1$ and $V_1$ in a natural fashion. For this reason, we dispense with the adjectives 'internal' and 'external' and use the notation $U \oplus V$ in both cases.

   (d) If $U$ and $V$ are finite-dimensional, verify that

$$dim(U \oplus V) = dim(U) + dim(V)$$

4. (This exercise is to lead up to a verification of the fact that the collection $\mathcal{W}$ of subspaces of a finite -dimensional vector space $V$ is a lattice.) Let $W_1, W_2 \in \mathcal{W}$.

   (a) Verify that $W_1 \cap W_2$ is also a subspace and contains any subspace that is contained in both $W_i, i = 1, 2$.

(b) Verify that $W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$ is also a subspace, and is contained in any subspace which contains both $W_i, i = 1, 2$.