Home-work 4

on lecture dated 26/09/09

1. Suppose V is an n-dimensional space. Show that the following conditions on a subset S of V with n elements are equivalent:
   (a) S is linearly independent
   (b) $sp(S) = V$
   (c) S is a basis for V
2. Show that $\{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$ is linearly independent and hence a basis for $\mathbb{R}^3$; and express a vector (a, b, c) as a linear combination of these vectors.
3. Find a basis of $\mathbb{R}^4$ which contains the set $\{(1, 1, -1, -1), (1, 1, 1, 1)\}$ (thus illustrating the fact that any linearly independent set can be extended to a basis).
4. Find a basis of $\mathbb{R}^2$ which is contained in the set $\{(1, 2), (3, 6), (3, 8)\}$ (thus illustrating the fact that any spanning set can be shrunk to a basis).
5. Show that the set $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 2x_1 + x_2 - x_3 = 0\}$ is a two-dimensional vector space
6. Prove that every finite-dimensional vector space V has a basis, by showing that, in fact, any linearly independent subset of V can be extended to a basis for V. Deduce in particular that if W is a subspace of V, then $dim(W) \leq dim(V)$.
7. Show that if a vector space has a spanning set with n elements, then it cannot contain a linearly independent set with $(n + 1)$ elements. In particular, deduce that a vector space has a finite spanning set if and only if it does not contain arbitrarily large linearly dependent sets.