Home-work 12

on lecture dated 12/12/09

1. If $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$ and if $\{e^1, e^2, e^3\}$ denotes the asociated dual basis for $(\mathbb{R}^3)^*$, show that

   (a) $\{e^i \otimes e^j : 1 \leq i, j \leq 3\}$ is a basis for the space $\mathcal{B}^2(\mathbb{R}^3)$ of all bilinear functions on $\mathbb{R}^3$, where $e^i \otimes e^j(x, y) = e^i(x)e^j(y)$ for all $x, y \in \mathbb{R}^3$.

   (b) With the foregoing notation, if we define, for $1 \leq i < j \leq 3$,
   $$e^i \wedge e^j = e^i \otimes e^j - e^j \otimes e^i \ ,$$
   show that $\{e^i \wedge e^j : 1 \leq i < j \leq 3\}$ is a basis for the space $\bigwedge^2 \mathbb{R}^3$ of bilinear forms ϕ on $\mathbb{R}^3$ which are alternating in the sense that $\phi(x, y) = -\phi(y, x) \ \forall x, y \in \mathbb{R}^3$.

2. (a) Show that the set $\{A \in M_n(\mathbb{R}) : A' = -A\}$ of *skew-symmetric* matrices is a vector subspace of $M_n(\mathbb{R})$ and show that its dimension is $\frac{n(n-1)}{2}$.

   (b) Show that the set $\mathcal{B}^2(\mathbb{R}^n)$ of bilinear functions on $\mathbb{R}^n$ is a vector space of dimension n^2, and that the set $\bigwedge^2 \mathbb{R}^n$ of alternating bilinear functions is a subspace.

   (c) Exhibit an isomorphism $\mu : \mathcal{B}^2(\mathbb{R}^n) \cong M_n(\mathbb{R})$ which maps $\bigwedge^2 \mathbb{R}^n$ onto the set of skew-symmetric $n \times n$ matrices, and thus deduce that $dim(\bigwedge^2 \mathbb{R}^n) = \frac{n(n-1)}{2}$.

3. Let A_n denotes the set of permutations which can be expressed as a product of an even number of transpositions. Assume that $A_n \neq S_n$ for $n > 1$.

   (a) If $\tau \in S_n \setminus A_n$, show that $\sigma \mapsto \tau\sigma$ is a bijective map of A_n onto $(S_n \setminus A_n)$, and that the equation
   $$\epsilon(\pi) = \left\{ \begin{array}{rl} 1 & \text{if } \pi \in A_n \\ -1 & \text{if } \pi \notin A_n \end{array} \right\} .$$
   defines a group homomorphism from S_n onto $\{\pm 1\}$.

   (b) Deduce that A_n is in fact a normal subgroup of S_n.

   (c) If $\tau_i, i = 1, 2$ are distinct transpositions, show that there exists a permutation σ such that $\tau_2 = \sigma\tau_1\sigma^{-1}$.

   (d) Deduce that no transposition can belong to A_n, and that no element of A_n can be expressed as a product of an odd number of transpositions.