

Cellularity and the Jones basic construction

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August, 2010

Introduction

The goal of this work is to study certain finite dimensional algebras that arise in invariant theory, knot theory, subfactors, QFT, and statistical mechanics.

The algebras in question have parameters; for generic values of the parameters, they are semisimple, but it is also interesting to study non-semisimple specializations. It turns out that operator algebra ideas — specifically, **the Jones basic construction** — are **still useful in the non-semisimple case**.

Introduction – cont.

More explicitly, we develop a framework for studying several important examples of **pairs of towers of algebras**,

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots \quad \text{and} \quad Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq \dots$$

such as

- $A_n =$ Brauer algebra, $Q_n =$ symmetric group algebra.
- $A_n =$ BMW algebra, $Q_n =$ Hecke algebra.
- $A_n =$ cyclotomic BMW algebra, $Q_n =$ cyclotomic Hecke algebra.
- $A_n =$ partition algebra, $Q_n =$ “stuttering” sequence of symmetric group algebras.
- etc.

Introduction, cont. 2

Some properties of these examples:

1. The algebras are **generically semisimple**, and in the generic semisimple case, the tower $(A_n)_{n \geq 0}$ is obtained from the tower $(Q_n)_{n \geq 0}$ “by repeated Jones basic constructions.”
2. More explicitly, this means that there is a generic (integral) ground ring R for A_n (independent of n). Every instance of A_n over a ground ring S is a specialization: $A_n^S = A_n^R \otimes_R S$. With F the field of fractions of R , the algebra A_n^F is semisimple.
3. A_n has an ideal J_n such that $Q_n \cong A_n/J_n$, and J_{n+1}^F is isomorphic to a Jones basic construction for the pair $A_{n-1}^F \subset A_n^F$.

Introduction, cont. 3

4. All this results from a method due to **Wenzl** in the 80's for showing the generic semisimplicity and determining the generic branching diagram of the tower $(A_n^F)_{n \geq 0}$. (Wenzl applied this to Brauer and BMW algebras.)
5. One is also interested in non-semisimple specializations (e.g. symmetric group in characteristic p , Hecke algebras at roots of unity), and here the **framework of cellularity, due to Graham and Lehrer**, is helpful.
6. For studying cellularity of the A_n , it suffices to work over the generic ground ring R , since cellularity is preserved under specialization.
7. In the examples, it is known that the cellular structures of the Q_n 's are *coherent*, which means well behaved with respect to induction and restriction.

Introduction, cont. 4

8. A reflection of this is that one has cellular bases indexed by paths on the generic branching diagram of $(Q_n)_{n \geq 0}$ that are well behaved with respect to restriction (**path bases**).
9. In our examples, cellularity of the algebras A_n was already known, but previous proofs generally did not give coherence of the cellular structures, nor path bases, or only gave such results by methods special to particular examples.

Introduction, cont. 5

In this work we have found a cellular analogue of Wenzl's construction which applies uniformly to all the examples, and produces coherent cellular structures and (cellular) path bases for the A_n 's.

Advantages:

- Easy to apply in examples, gives efficient proof of cellularity.
- Considerable simplification of previous work in the case of cyclotomic BMW algebras.
- Relates cellular structure to Jones basic construction: Cell modules of A_n either "come from" Q_n or from A_{n-2} .
- Produces path bases of A_n
- One can also give a criterion for lifting Jucys–Murphy elements from Q_n to A_n .

Cellularity, a tool for the non-semisimple theory

Next, we want to introduce a tool for studying the non-semisimple case, namely the theory of *cellularity*, due to Graham and Lehrer.

What is cellularity?

Let A be an algebra with involution $*$ over an integral domain S . A is said to be cellular if there exists a finite partially ordered set (Λ, \geq) and for each $\lambda \in \Lambda$, a finite index set $\mathcal{T}(\lambda)$, such that

- A has an S -basis $\{c_{s,t}^\lambda : \lambda \in \Lambda; s, t \in \mathcal{T}(\lambda)\}$.
- For every order ideal Γ of Λ ,

$$A(\Gamma) := \text{span}\{c_{s,t}^\lambda : \lambda \in \Gamma, s, t \in \mathcal{T}(\lambda)\}$$

is a $*$ -ideal of A . In particular, write A^λ for $A(\{\mu : \mu \geq \lambda\})$ and \check{A}^λ for $A(\{\mu : \mu > \lambda\})$.

- $(c_{s,t}^\lambda)^* \equiv c_{t,s}^\lambda$ modulo \check{A}^λ .
- For each $\lambda \in \Lambda$, there is an A -module Δ^λ , free as S -module, with basis $\{c_t^\lambda : t \in \mathcal{T}(\lambda)\}$, such that the map $\alpha^\lambda : A^\lambda / \check{A}^\lambda \rightarrow \Delta^\lambda \otimes_R (\Delta^\lambda)^*$ defined by $\alpha^\lambda : c_{s,t}^\lambda + \check{A}^\lambda \mapsto c_s \otimes (c_t^\lambda)^*$ is an A - A bimodule isomorphism.

What is cellularity?, cont.

- This whole apparatus is called a cell datum. The R -basis is called a cellular basis.
- The modules Δ^λ are called *cell modules*. When the ground ring is a field, and the algebra A is semisimple, these are exactly the simple modules.
- In general, Δ^λ has a canonical bilinear form. With $\text{rad}(\lambda)$ the radical of this form, and with the ground ring a field, $\Delta^\lambda/\text{rad}(\lambda)$ is either zero or simple, and all simples are of this form.

Cellularity – Example, the Hecke algebras

Definition 1

The Hecke algebra $H_n^S(q^2)$ over S is the quotient of the braid group algebra over S by the Hecke skein relation:

$$\begin{array}{c} \diagup \\ \diagdown \end{array} - \begin{array}{c} \diagdown \\ \diagup \end{array} = (q - q^{-1}) \begin{array}{c} | \\ | \end{array}.$$

Fact:

The Hecke algebras $H_n^S(q^2)$ are cellular, with $\Lambda = Y_n$, the set of Young diagrams with n boxes, ordered by *dominance*, and $\mathcal{T}(\lambda)$ the set of standard tableaux of shape λ . The cell modules are known as *Specht modules*. The cellular structure is due to Murphy. See, for example, A. Mathas, *Iwahori-Hecke Algebras and Schur Algebras of the Symmetric Group*, AMS University Lecture Series.

Cellularity – Basis free formulation

Definition 2

A Λ -cell net is an (order preserving) map from the set of order ideals of Λ to the set of $*$ -ideals of A , $\Gamma \mapsto A_\Gamma$, with several natural properties, the most important being:

For each $\lambda \in \Lambda$, there is an A -module M^λ , finitely generated and free as an S -module, such that whenever $\Gamma \subseteq \Gamma'$ are order ideals of Λ , with $\Gamma' \setminus \Gamma = \{\lambda\}$, then there exists an isomorphism of A - A -bimodules

$$\alpha : A_{\Gamma'} / A_\Gamma \rightarrow M^\lambda \otimes_S i(M^\lambda),$$

satisfying $i \circ \alpha = \alpha \circ i$, where i is the map, induced from $*$, which interchanges left and right A -modules.

Cellularity – basis free formulation – continued

Proposition 3

Let A be an R -algebra with involution, and let (Λ, \geq) be a finite partially ordered set. Then A has a cell datum with partially ordered set Λ if, and only if, A has a Λ -cell net.

Of course, then M^λ will turn out to be the cell module corresponding to λ .

Coherence – Introduction

It is a general principle that representation theories of the Hecke algebras $H_n(q)$ or of the symmetric group algebras KS_n should be considered all together, that induction/restriction between H_n and H_{n-1} plays a role in building up the representation theory.

Coherence of cellular structures is the cellular version of this principle.

Coherence

Definition 4

A sequence $(A_n)_{n \geq 0}$ of cellular algebras, with cell data (Λ_n, \dots) is coherent if for each $\mu \in \Lambda_n$, the restriction of Δ^μ to A_{n-1} has a filtration

$$\text{Res}(\Delta^\lambda) = F_t \supseteq F_{t-1} \supseteq \cdots \supseteq F_0 = (0),$$

with $F_j/F_{j-1} \cong \Delta^{\lambda_j}$ for some $\lambda_j \in \Lambda_{n-1}$, and similarly for induced modules.

The sequence $(A_n)_{n \geq 0}$ is strongly coherent if, in addition,

$$\lambda_t < \lambda_{t-1} < \cdots < \lambda_1$$

in Λ_{n-1} , and similarly for induced modules.

Strong coherence

Given a strongly coherent sequence $(A_n)_{n \geq 0}$ of cellular algebras, with mild additional assumptions, satisfied in our examples, one always has *path bases* of cell modules and of the algebras themselves.

Example of strong coherence: The sequence of Hecke algebras $H_n(q)$ is a strongly coherent sequence of cellular algebras. This results from combining theorems of Murphy, Dipper-James, and Jost from the 80's

Main theorem

Theorem 5

Suppose $(A_n)_{n \geq 0}$ and $(Q_n)_{n \geq 0}$ are two sequence of $*$ - algebras over R . Let F be the field of fractions of R . Assume:

1. $(Q_n)_{n \geq 0}$ is a (strongly) coherent tower of cellular algebras.
2. $A_0 = Q_0 = R, A_1 \cong Q_1$.
3. For each $n \geq 2$, A_n has an essential idempotent $e_{n-1} = e_{n-1}^*$ and

$$0 \rightarrow A_n e_{n-1} A_n \rightarrow A_n \rightarrow Q_n \rightarrow 0,$$

is a short exact sequence of $*$ - algebras.

4. $A_n^F = A_n \otimes_R F$ is split semisimple.
5. (a) e_{n-1} commutes with A_{n-1} , and $e_{n-1} A_{n-1} e_{n-1} \subseteq A_{n-2} e_{n-1}$,
(b) $A_n e_{n-1} = A_{n-1} e_{n-1}$ and $x \mapsto x e_{n-1}$ is injective from A_{n-1} to $A_{n-1} e_{n-1}$, and (c) $e_{n-1} = e_{n-1} e_n e_{n-1}$.

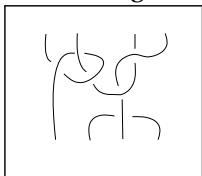
Then $(A_n)_{n \geq 0}$ is a (strongly) coherent tower of cellular algebras.

The BMW algebras

A chief example is $A_n = \text{BMW}$ (Birman-Murakami-Wenzl) algebra on n strands, and $Q_n = \text{Hecke}$ algebra on n strands.

The BMW algebra is an algebra of braid like objects, namely *framed*

(n, n) -tangles:



Tangles can be represented by quasi-planar diagrams as shown here. Tangles are multiplied by stacking (like braids).

Definition of BMW algebras

Definition 6

Let S be a commutative unital ring with invertible elements ρ, q, δ satisfying $\rho^{-1} - \rho = (q^{-1} - q)(\delta - 1)$. The BMW algebra W_n^S is the S -algebra of framed (n, n) -tangles, modulo the Kauffman skein relations:

1. (Crossing relation) $\begin{array}{c} \diagup \\ \diagdown \end{array} - \begin{array}{c} \diagdown \\ \diagup \end{array} = (q^{-1} - q) \left(\begin{array}{c} \diagup \\ \diagup \end{array} - \begin{array}{c} \diagdown \\ \diagdown \end{array} \right)$.
2. (Untwisting relation) $\begin{array}{c} \diagup \\ \diagdown \end{array} = \rho \mid$ and $\begin{array}{c} \diagdown \\ \diagup \end{array} = \rho^{-1} \mid$.
3. (Free loop relation) $T \cup \bigcirc = \delta T$, where $T \cup \bigcirc$ is the union of a tangle T and an additional closed loop with zero framing.

The BMW algebras, cont.

- W_n^S imbeds in W_{n+1}^S . On the level of tangle diagrams, the embedding is by adding one strand on the right.
- The BMW algebras have an S -linear algebra involution, acting by turning tangle diagrams upside down.
- The following tangles generate the BMW algebra

$$e_i = \begin{array}{|c|} \hline \begin{array}{c} \text{Diagram of } e_i: \text{Two vertical strands labeled } i \text{ and } i+1. \text{ The strand } i \text{ has a loop that passes over strand } i+1. \end{array} \\ \hline \end{array}$$

and

$$g_i = \begin{array}{|c|} \hline \begin{array}{c} \text{Diagram of } g_i: \text{Two vertical strands labeled } i \text{ and } i+1. \text{ Strand } i \text{ crosses over strand } i+1. \end{array} \\ \hline \end{array} .$$

The element e_i is an essential idempotent with $e_i^2 = \delta e_i$.
One has $e_i e_{i\pm 1} e_i = e_i$.

The BMW algebras, cont. 2

The ideal J_n generated by one or all e_i 's in W_n^S satisfies $W_n^S/J_n \cong H_n^S(q^2)$ where $H_n^S(q^2)$ is the Hecke algebra.

There is a generic ground ring for the BMW algebras, namely

$$R = \mathbb{Z}[\mathbf{q}^{\pm 1}, \mathbf{\rho}^{\pm 1}, \mathbf{\delta}^{\pm 1}]/J,$$

where J is the ideal generated by

$$\mathbf{\rho}^{-1} - \mathbf{\rho} - (\mathbf{q}^{-1} - \mathbf{q})(\mathbf{\delta} - 1)$$

and where the bold symbols denote indeterminants.

BMW – Generic ground ring, cont. 3

The generic ground ring R is an integral domain, with field of fractions $F = \mathbb{Q}(\mathbf{q}, \boldsymbol{\rho})$, and $\boldsymbol{\delta} = (\boldsymbol{\rho}^{-1} - \boldsymbol{\rho})/(\mathbf{q}^{-1} - \mathbf{q}) + 1$ in F .

For every instance of the BMW over a ring S with parameters $\boldsymbol{\rho}, \mathbf{q}, \boldsymbol{\delta}$, one has $W_n^S \cong W_n^R \otimes_R S$.

The BMW algebras over F are semisimple (theorem of Wenzl).

Outline of proof: Assume W_k^F is s.s. for $k \leq n$. The ideal $J_{n+1} \subseteq W_{n+1}^F$ is the Jones basic construction for $W_{n-1}^F \subseteq W_n^F$, so is also s.s. The quotient W_{n+1}^F/J_{n+1} is the Hecke algebra $H_n^F(\mathbf{q}^2)$, which is s.s. since \mathbf{q} is not a root of 1. Hence W_{n+1}^F is s.s.

The BMW algebras, application of our theorem

Now let's see what's involved in applying the theorem to the BMW algebras (with $A_n = W_n^R$, and $Q_n = H_n^R(q^2)$). Hypothesis (1) is the (strong) coherence of the sequence of Hecke algebras, which is a significant theorem about Hecke algebras. Hypothesis (4) on the semisimplicity of W_n^F is Wenzl's theorem. Everything else is elementary, and already contained in Birman-Wenzl.

All the other examples work pretty much the same way.

Some idea of the proof

The proof is inductive and is a cellular version of Wenzl's semisimplicity proof. Suppose we know that A_k is cellular (and satisfies all the conclusions of the theorem) for $k \leq n$.

Then we want to show the same for A_{n+1} . The main point is to show that $J_{n+1} = A_{n+1}e_nA_{n+1} = A_n e_n A_n$ is a "cellular ideal" in A_{n+1} . (This suffices to show cellularity, because we also have that $A_{n+1}/J_{n+1} \cong Q_{n+1}$ is cellular by hypothesis, and extensions of cellular algebras by cellular ideals are also cellular.)

Some idea of the proof–cont.

We have a Λ_{n-1} –cell net $\Gamma \mapsto J(\Gamma) := \text{span}\{c_{\mathfrak{s},\mathfrak{t}}^\lambda : \lambda \in \Gamma, \mathfrak{s}, \mathfrak{t} \in \mathcal{T}(\lambda)\}$. Now we want to show that

$$\Gamma \mapsto \hat{J}(\Gamma) := A_n e_n J(\Gamma) A_n$$

is a Λ_{n-1} –cell net in $A_n e_n A_n$. Along the way to doing this we also have to show that

$$J'(\Gamma) := A_n \otimes_{A_{n-1}} J(\Gamma) \otimes_{A_{n-1}} A_n \cong A_n e_n J(\Gamma) A_n$$

and in particular $A_n \otimes_{A_{n-1}} A_n \cong A_n e_n A_n$, and if $\Gamma_1 \subseteq \Gamma_2$, then also $J'(\Gamma_1)$ imbeds in $J'(\Gamma_2)$. This is a bit tricky because $A_n e_n A_n$ is not a unital algebra and A_n is not projective as A_{n-1} –module.

Last slide!

Now if $\lambda \in \Lambda_{n-1}$ and $\Gamma_1 \subseteq \Gamma_2$, with $\Gamma_2 \setminus \Gamma_1 = \lambda$, then

$$\begin{aligned} \hat{J}(\Gamma_2)/\hat{J}(\Gamma_1) &\cong J'(\Gamma_2)/J'(\Gamma_1) \\ &\cong A_n \otimes_{A_{n-1}} J(\Gamma_1)/J(\Gamma_2) \otimes_{A_{n-1}} A_n \\ &\cong A_n \otimes_{A_{n-1}} (\Delta^\lambda \otimes_R (\Delta^\lambda)^*) \otimes_{A_{n-1}} A_n \\ &\cong (A_n \otimes_{A_{n-1}} \Delta^\lambda) \otimes_R (\Delta^\lambda)^* \otimes_{A_{n-1}} A_n \end{aligned}$$

Now we need that $M^\lambda = A_n \otimes_{A_{n-1}} \Delta^\lambda$ is free as R -module, to verify the crucial property in the definition of a cell net. But as A_n -module, M^λ is $\text{Ind}(\Delta^\lambda)$, and by an induction assumption on coherence of the cellular structures on $(A_k)_{k \leq n}$, this has a filtration by cell modules for A_n , so is free as an R -module.