VON NEUMANN ALGEBRAS: INTRODUCTION, MODULAR THEORY AND CLASSIFICATION THEORY

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Introduction:

von Neumann algebras, as they are called now, first made their appearance under the name *Rings of Operators* in a series of seminal papers - see [MvN1], [MvN2], [vN1], [MvN3] - by F.J. Murray and J. von Neumann starting in 1936. In their first paper [MvN1], they specifically cite 'attempts to generalize the theory of unitary group representations' and 'demands by various aspects of the quantum mechanical formalism' among the reasons for the elucidation of this subject.

In fact, the simplest definition of a von Neumann algebra is via unitary group representations: a collection M of continuous linear operators on a Hilbert space[1] $\mathcal{H}$ is a von Neumann algebra precisely when there is a representation ρ of a group G as unitary operators on $\mathcal{H}$ such that

$$M = \{x \in \mathcal{L}(\mathcal{H}) : x\rho(t) = \rho(t)x \ \forall t \in G\}$$

As above, we shall write $\mathcal{L}(\mathcal{H})$ for the collection of all continuous linear operators on the Hilbert space $\mathcal{H}$; recall that a linear mapping $x : \mathcal{H} \to \mathcal{H}$ is continuous precisely when there exists a positive constant K such that $\|x\xi\| \leq K\|\xi\| \ \forall \xi \in \mathcal{H}$. If the norm $\|x\|$ of the operator x is defined as the smallest constant K with the above property, then the set $\mathcal{L}(\mathcal{H})$ acquires the structure of a Banach space. In fact $\mathcal{L}(\mathcal{H})$ is a Banach *-algebra with respect to the composition product, and involution $x \mapsto x^*$ given by

$$\langle x\xi, \eta\rangle = \langle \xi, x^*\eta\rangle \ \forall \xi, \eta \in \mathcal{H} \ .$$

The first major result in the subject is the remarkable *double commutant theorem*, which establishes the equivalence of a purely algebraic requirement to purely topological ones. We need two bits of terminology to be able to state the theorem.

First, define the *commutant* S' of a subset $S \subset \mathcal{L}(\mathcal{H})$ by

$$S' = \{x' \in \mathcal{L}(\mathcal{H}) : x'x = xx' \ \forall x \in S\} \ .$$

Second, the *strong (*resp. *weak) operator topology* is the topology on $\mathcal{L}(\mathcal{H})$ of 'pointwise strong (resp., weak) convergence': i.e., $x_n \to x$ precisely when $\|x_n\xi - x\xi\| \to 0 \ \forall \xi \in \mathcal{H}$ (resp., $\langle x_n\xi - x\xi, \eta\rangle \to 0 \ \forall \xi, \eta \in \mathcal{H}$).

Theorem 1. *The following conditions on a unital *-subalgebra M of $\mathcal{L}(\mathcal{H})$ are equivalent:*

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[1] In order to avoid some potential technical problems, we shall restrict ourselves to **separable** Hilbert spaces throughout these notes.

(1) $M = M''(= (M')')$.
(2) *M is closed in the strong operator topology.*
(3) *M is closed in the weak operator topology.*

The conventional definition of a von Neumann algebra is that it is a unital *-subalgebra of $\mathcal{L}(\mathcal{H})$ which satisfies the equivalent conditions above. The equivalence with our earlier 'simplest definition' is a consequence of the double commutant theorem and the fact that any element of a von Neumann algebra is a linear combination of four unitary elements of the algebra: simply take G to be the group of unitary operators in M'.

Another consequence of the double commutant theorem is that von Neumann algebras are closed under any 'canonical construction'. For instance, the uniqueness of the spectral measure $E \mapsto P_x(E)$ associated to a normal operator x shows that if u is unitary, then $P_{uxu^*}(E) = uP_x(E)u^*$ for all Borel sets E. In particular, if $x \in M$ and $u' \in \mathcal{U}(M')$, then $u'P_x(E)u'^* = P_{u'xu'^*}(E) = P_x(E)$, and hence, we may conclude that $P_x(E) \in \mathcal{U}(M')' = (M')' = M$;[2] i.e., if a von Neumann algebra contains a normal operator, it also contains all the associated spectral projections. This fact, together with the spectral theorem has the consequence that any von Neumann algebra M is the closed linear span of $\mathcal{P}(M)$.

The analogy with unitary group representations is fruitful. Suppose then that $M = \rho(G)'$, for a unitary representation of G. Then the last sentence of the previous paragraph implies that $\rho(G)' = \mathbb{C}$ precisely when there exist no non-trivial ρ-stable subspaces,[3] i.e., when ρ is irreducible. In general the ρ-stable subspaces are precisely the ranges of projection operators in M. The notion of unitary equivalence of subrepresentations of ρ is seen to translate to the equivalence defined on the set $\mathcal{P}(M)$ of projections in M, whereby $p \sim q$ if and only if there exists an operator $u \in M$ such that $u^*u = p$ and $uu^* = q$. (Such a u is called a partial isometry, with 'initial space' = range p, and 'final space' = range q.) This is the definition of what is known as the *Murray-von Neumann equivalence rel* M and is denoted by $\sim_M$. The following accompanying definition is natural: if $p, q \in \mathcal{P}(M)$, say $p \preceq_M q$ if there exists $p_0 \in \mathcal{P}(M)$ such that $p \sim_M p_0 \leq q$ - where of course $e \leq f \Leftrightarrow range(e) \subset range(f)$.

The Murray-von Neumann classification of factors:

We start with a fact (whose proof is quite easy) and a consequent fundamental definition.

Proposition 2. *The following conditions on a von Neumann algebra M are equivalent:*

(1) *for any $p, q \in \mathcal{P}(M)$, it is true that either $p \preceq_M q$ or $q \preceq_M p$.*
(2) $Z(M) = M \cap M' = \mathbb{C}$.

The von Neumann algebra M is called a **factor** *if it satisfies the equivalent conditions above.*

[2]We will write $\mathcal{U}(N)$ (resp., $\mathcal{P}(N)$) to denote the collection of unitary (resp., projection) operators in any von Neumann algebra N.

[3]Here and in the sequel, we identify $\mathbb{C}$ with its image under the unique unital homomorphism of $\mathbb{C}$ into $\mathcal{L}(\mathcal{H})$; and we reserve the symbol $Z(M)$ to denote the center of M.

The alert reader would have noticed that if G is a finite group, then $\rho(G)'$ is a factor precisely when the representation ρ is 'isotypical'. Thus, the 'representation theoretic fact' that any unitary representation is expressible as a direct sum of isotypical subrepresentations, translates into the 'von Neumann algebraic fact' that any $*$-subalgebra of $\mathcal{L}(\mathcal{H})$ is isomorphic, when $\mathcal{H}$ is finite-dimensional, to a direct sum of factors. In complete generality, von Neumann showed (cf. [vN2]) that any von Neumann algebra is expressible as a 'direct integral of factors'. We shall interpret this fact from 'reduction theory' as the statement that all the magic/mystery of von Neumann algebras is contained in factors and hence restrict ourselves, for a while, to the consideration of factors.

Murray and von Neumann initiated the study of a general factor M via a qualitative as well as a quantitative analysis of the relation $\preceq_M$ on $\mathcal{P}(M)$. First, call a $p \in \mathcal{P}(M)$ infinite if there exists a $p_0 \leq p$ such that $p \sim_M p_0$ and $p_0 \neq p$; otherwise, say p is finite. They obtain an analogue, called the *dimension function*, of the Haar measure, as follows.

Theorem 3. *(a) With M as above, there exists a function $D_M : \mathcal{P}(M) \to [0, \infty]$ which satisfies the following properties, and is determined up to a multiplicative constant, by them:*

- $p \preceq_M q \Leftrightarrow D_M(p) \leq D_M(q)$
- *p is finite if and only if $D_M(p) < \infty$*
- *If $\{p_n : n = 1, 2, \cdots\}$ is any sequence of pairwise orthogonal projections in $\mathcal{P}(M)$ and $p = \sum_n p_n$, then $D_M(p) = \sum_n D_M(p_n)$*

(b) M falls into exactly one of five possible cases, depending on which of the following sets is the range of some scaling of D_M:

- *(I_n)* $\{0, 1, 2, \cdots, n\}$
- *(I_∞)* $\{0, 1, 2, \cdots, \infty\}$
- *(II_1)* $[0, 1]$
- *(II_∞)* $[0, \infty]$
- *(III)* $\{0, \infty\}$

In words, we may say that a factor M is of:

(1) type I (i.e., of type I_n for some $1 \leq n \leq \infty$) precisely when M contains a minimal projection
(2) type II (i.e., of type II_1 or II_∞) precisely when M contains non-zero finite projections but no minimal projections
(3) type III precisely when M contains no non-zero finite projections.

Examples:

$L^\infty(\Omega, \mu)$ may be regarded as a von Neumann algebra acting on $L^2(\Omega, \mu)$ as multiplication operators; thus, if we set $m_f(\xi) = f\xi$, then $m : f \mapsto m_f$ defines an isomorphism of $L^\infty(\Omega, \mu)$ onto a commutative von Neumann subalgebra of $\mathcal{L}(L^2(\Omega, \mu))$. In fact, 'up to multiplicity', this is how any commutative von Neumann algebra looks.

It is a simple exercise to prove that $M \subset \mathcal{L}(\mathcal{H})$ is a factor of type $I_n, 1 \leq n \leq \infty$ if and only if there exist Hilbert spaces $\mathcal{H}_n$ and $\mathcal{K}$ and identifications $\mathcal{H} = \mathcal{H}_n \otimes \mathcal{K}$, $M = \{x \otimes id_\mathcal{K} : x \in \mathcal{L}(\mathcal{H}_n)\}$ where dim $\mathcal{H}_n = n$; and so $M \cong \mathcal{L}(\mathcal{H}_n)$.

To discuss examples of the other types, it will be convenient to use 'crossed products' of von Neumann algebras by ergodically acting groups of automorphisms. We shall now digress with a discussion of this generalization of the notion of a semi-direct product of groups.

If $\alpha : G \to Aut(M)$ is an action of a countable group G on M, where $M \subset \mathcal{L}(\mathcal{H})$ is a von Neumann algebra, and $\widetilde{\mathcal{H}} = \ell^2(G, \mathcal{H})$, there are representations $\pi : M \to \mathcal{L}(\widetilde{\mathcal{H}})$ and $\lambda : G \to \mathcal{U}(\mathcal{L}(\widetilde{\mathcal{H}}))$ defined by

$$(\pi(x)\xi)(s) = \alpha_{s^{-1}}(x)\xi(s), (\lambda(t)\xi)(s) = \xi(t^{-1}s) .$$

These representations satisfy the commutation relation $\lambda(t)\pi(x)\lambda(t^{-1}) = \pi(\alpha_t(x))$, and the crossed-product $M \rtimes_\alpha G$ is the von Neumann subalgebra of $\mathcal{L}(\widetilde{\mathcal{H}})$ defined by $\widetilde{M} = (\pi(M) \cup \lambda(G))''$.

Let us restrict ourselves to the case of $M = L^\infty(\Omega, \mu)$ acting on $L^2(\Omega, \mu)$. In this case, it is true that any automorphism of M is of the form $f \mapsto f \circ T^{-1}$, where T is a 'non-singular transformation of the measure space (Ω, μ)' (= a bijection which preserves the class of sets of μ-measure 0). So, an action of G on M is of the form $\alpha_t(f) = f \circ T_t^{-1}$, for some homomorphism $t \mapsto T_t$ from G to the group of non-singular transformations of (Ω, μ). We have the following elegantly complete result from [MvN1].

Theorem 4. *Let M, G, α be as in the last paragraph, and let $\widetilde{M} = M \rtimes_\alpha G$. Assume the G-action is 'free', meaning that if $t \neq 1 \in G$, then $\mu(\{\omega \in \Omega : T_t(\omega) = \omega\}) = 0$. Then:*

(1) *$\widetilde{M}$ is a factor if and only if G acts ergodically on (Ω, μ) - meaning that the only G-fixed functions in M are the constants.*
(2) *Assume that G acts ergodically. then the type of the factor $\widetilde{M}$ is determined as follows:*
   - *$\widetilde{M}$ is of type I or II if and only if there exists a G-invariant measure ν which is mutually absolutely continuous with respect to μ, meaning $\nu(E) = 0 \Leftrightarrow \mu(E) = 0$; (the ergodicity assumption implies that such a ν is necessarily unique up to scaling by a positive constant;)*
   - *$\widetilde{M}$ is of type I_n precisely when the ν as above is totally atomic, and Ω is the disjoint union of n atoms for ν;*
   - *$\widetilde{M}$ is of type II precisely when the ν as above is non-atomic;*
   - *$\widetilde{M}$ is of finite type - meaning that 1 is a finite projection in $\widetilde{M}$ - precisely when the ν as above is a finite measure;*
   - *$\widetilde{M}$ is of type III if and only if there exists no ν as above.*

Thus, we get all the types of factors by this construction; for instance, we may take:

(I_n) $G = \mathbb{Z}_n$ acting on $\Omega = \mathbb{Z}_n$ by translation, and $\mu = \nu =$ counting measure

(I_∞) $G = \mathbb{Z}$ acting on $\Omega = \mathbb{Z}$ by translation, and $\mu = \nu =$ counting measure

(II_1) $G = \mathbb{Z}$ acting on $\Omega = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ by powers of an aperiodic rotation, and $\mu = \nu =$ arc-length measure

(II_∞) $G = \mathbb{Q}$ acting on $\Omega = \mathbb{R}$ by translations, and $\mu = \nu =$ Lebesgue measure

(III) $G = ax + b$ group acting in the obvious manner on $\Omega = \mathbb{R}$, $\mu = \nu =$ Lebesgue measure.

Such crossed products of a commutative von Neumann algebra by an ergodically acting countable group were intensively studied by Krieger ([Kri1], [Kri1]). We shall simply refer to such factors as *Krieger factors*.[4]

Abstract von Neumann algebras:

So far, we have described matters as they were in von Neumann's time. To come to the modern era, it is desirable to 'free a von Neumann algebra from the ambient Hilbert space' and to regard it as an abstract object in its own right which can act on different Hilbert spaces - eg: $L^\infty(\Omega,\mu)$ is an object worthy of study in its own right, without reference to $L^2(\Omega,\mu)$.

The abstract viewpoint is furnished by a theorem of Sakai(cf. [Sak]); let us define an *abstract von Neumann algebra* to be an abstract C^*-algebra[5] M which admits a pre-dual M_* (i.e., M is isometrically isomorphic to the Banach dual space $(M_*)^*$. It turns out that a pre-dual of such an abstract von Neumann algebra is unique up to isometric isomorphism. Consequently, an abstract von Neumann algebra comes equipped with a canonical 'weak*-topology', usually called the *σ-weak topology* on M. The natural morphisms in the category of abstract von Neumann algebras are $*$-homomorphisms which are continuous with respect to σ-weak topologies on domain and range. It is customary to call a linear map between abstract von Neumann algebras *normal* if it is continuous with respect to σ-weak topologies on domain and range.

The equivalence of the 'abstract' definition of this section, with the 'concrete' one given earlier (which depends on an ambient Hilbert space) relies on the following four facts:

(1) $\mathcal{L}(\mathcal{H})$ is an abstract von Neumann algebra, with the pre-dual $\mathcal{L}(\mathcal{H})_*$ being the so-called 'trace class' of operators, equipped with the 'trace-norm'.
(2) A self-adjoint subalgebra of $\mathcal{L}(\mathcal{H})$ is closed in the strong operator topology, and is hence a 'concrete von Neumann algebra' precisely when it is closed in the σ-weak topology on $\mathcal{L}(\mathcal{H})$.
(3) If M is an abstract von Neumann algebra, and N is a *-subalgebra of M which is closed in the σ-weak topology of M, then N is also an abstract von Neumann algebra, with one candidate for N_* being $M_*/N_\perp$ (where $N_\perp = \{\rho \in M_* : n(\rho) = 0 \ \forall n \in N)$.
(4) Any abstract von Neumann algebra (with separable pre-dual) is isomorphic (in the category of abstract von Neumann algebras) to a (concrete) von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$ (for a separable $\mathcal{H}$).

With the abstract viewpoint available, we shall look for modules over a von Neumann algebra M, meaning pairs $(\mathcal{H},\pi)$ where $\pi : M \to \mathcal{L}(\mathcal{H})$ is a normal *-homomorphism.

A brief digression into the proof of fact (4) above - which asserts the existence of faithful M-modules - will be instructive and useful. Suppose M is an abstract von Neumann algebra. A linear functional ϕ on M is called a *normal state* if:

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[4]The term 'Krieger factor' is actually used for factors obtained from a slightly more general construction, with ergodic group actions replaced by more general ergodic equivalence relations. Since there is no difference in the two notions at least in good (amenable) cases, we will say no more about this.

[5]This is a Banach algebra with an involution related to the norm by the so-called C^*-identity $\|x\|^2 = \|x^*x\|$.

- (positivity) $\phi(x^*x) \geq 0 \; \forall x \in M$;
- (normality) $\phi : M \to \mathbb{C}$ is normal; and
- (normalization) $\phi(1) = 1$.

(Normal states on $L^\infty(\Omega, \mu)$ correspond to non-negative probability measures on Ω which are absolutely continuous with respect to μ.) It is true that there exist plenty of normal states on M. In fact, they linearly span M_*. This implies that if M_* is separable, then there exist normal states on M which are even *faithful* - meaning $\phi(x^*x) = 0 \Leftrightarrow x = 0$.

Fix a faithful normal state ϕ on M.[6] The well-known 'Gelfand-Naimark-Segal' construction then yields a faithful M-module which is usually denoted by $L^2(M, \phi)$ - motivated by the fact that if $M = L^\infty(\Omega, \mu)$, and $\phi(f) = \int f d\nu$, with ν a probability measure mutually absolutely continuous with respect to μ, then $L^2(M, \phi) = L^2(\Omega, \nu)$ with $L^\infty(\Omega, \mu)$ acting as multiplication operators. The construction mimics this case: the assumptions on ϕ ensure that the equation

$$\langle x, y \rangle = \phi(y^*x)$$

defines a positive-definite inner-product on M; let $L^2(M, \phi)$ be the Hilbert-space completion of M. It turns out that the operator of left-multiplication by an element of M extends as a bounded operator to $L^2(M, \phi)$, and it then follows easily that $L^2(M, \phi)$ is indeed a faithful M-module., thereby establishing fact (4) above.

Since we wish to distinguish between elements of the dense subspace M of $L^2(M, \phi)$ and the operators of left-multiplication by members of M, let us write $\hat{x}$ for an element of M when thought of as an element of $L^2(M, \phi)$, and x for the operator of left multiplication by x; thus, for instance, $\hat{x} = x\hat{1}$, and $x\hat{y} = \widehat{xy}$, $\langle x\hat{1}, \hat{1} \rangle = \phi(x)$, etc.

Modular theory:

While type III factors were more or less an enigma at the time of von Neumann, all that changed with the advent of Connes. The first major result of this 'type III era' is the celebrated 'Tomita-Takesaki theorem' (cf. [Tak]), which views the adjoint mapping on M as an appropriate operator on $L^2(M, \phi)$, and analyses its polar decomposition. Specifically, we have:

Theorem 5. *If ϕ is any faithful normal state on M, consider the densely defined conjugate linear operator given, with domain $\{\hat{x} : x \in M\}$, by $S_\phi^{(0)}(\hat{x}) = \widehat{x^*}$. Then*

1. *there is a unique conjugate-linear operator S_ϕ (the 'closure of $S_\phi^{(0)}$') whose graph is the closure of the graph of $S_\phi^{(0)}$; if we write $S_\phi = J_\phi \Delta_\phi^{\frac{1}{2}}$ for the polar decomposition of the conjugate-linear closed operator S_ϕ, then*
2. *J_ϕ is an antiunitary involution on $L^2(M, \phi)$ (i.e., it is a conjugate-linear norm-preserving bijection of $L^2(M, \phi)$ onto itself which is its own inverse)*
3. *Δ_ϕ is an injective positive self-adjoint operator on $L^2(M, \phi)$ such that $J_\phi f(\Delta_\phi) J_\phi = \bar{f}(\Delta_\phi^{-1})$ for all Borel functions $f : \mathbb{R} \to \mathbb{R}$, and most crucially*
4. $$J_\phi M J_\phi = M' \text{ and } \Delta_\phi^{it} M \Delta_\phi^{-it} = M \; \forall t \in \mathbb{R} \; .$$

[6] Consistent with our convention about separable $\mathcal{H}$'s, we shall only consider M's with separable pre-duals.

(Here and elsewhere, we shall identify $x \in M$ with the operator of 'left multiplication by x' on $L^2(M, \phi)$.)

Thus, each faithful normal state ϕ on M yields a one-parameter group $\{\sigma_t^\phi : t \in \mathbb{R}\}$ of M - referred to as the group of *modular automorphisms* - given by

$$\sigma_t^\phi(x) = \Delta_\phi^{it} x \Delta_\phi^{-it} .$$

The extent of dependence of the modular group on the state is captured precisely by Connes' Radon Nikodym theorem ([Con1]) which shows that the modular groups associated to two different faithful normal states are related by a 'unitary cocycle in M'. This has the consequence that if $\epsilon : Aut(M) \to Out(M) = Aut(M)/Int(M)$ is the quotient mapping - where $Int(M)$ denotes the normal subgroup of inner automorphisms given by unitary elements of M - then the one-parameter subgroup $\{\epsilon(\sigma_t^\phi) : t \in \mathbb{R}\}$ of $Out(M)$ is independent of ϕ.

Connes' classification and injective factors:

Given a factor M, Connes defined

$$S(M) = \bigcap \{spec(\Delta_\phi) : \phi \text{ a faithful normal state on } M\}$$

which is obviously an isomorphism invariant. He then classified ([Con1]) type III factors into a continuum of factors:

Theorem 6. *Let M be a factor. Then,*

1. $0 \in S(M) \Leftrightarrow M$ *is of type III; and*
2. *if M is a type III factor, there are three mutually exclusive and exhaustive possibilities:*
   - (III_0) $S(M) = \{0, 1\}$
   - (III_λ) $S(M) = \{0\} \cup \lambda^{\mathbb{Z}}$, *for some* $0 < \lambda < 1$
   - (III_1) $S(M) = [0, \infty)$

Example 7. Consider the compact group $\Omega = \prod_{n=1}^\infty G_n$ where G_n is a finite cyclic group of order ν_n for each n. Let $\mu = \prod_{n=1}^\infty \mu_n$, where μ_n is a probability measure defined on the subsets of G_n which assigns positive mass to each singleton. Let $G = \oplus_{n=1}^\infty G_n$ be the dense subgroup of Ω consisting of finitely non-zero sequences. It is not hard to see that each translation $T_g, g \in G$ (given by $T_g(\omega) = g + \omega$) is a non-singular transformation of the measure space (Ω, μ). The density of G in Ω shows that this action of G on $L^\infty(\Omega, \mu)$ is fixed-point-free and ergodic, with the result that the crossed-product $L^\infty(\Omega, \mu) \rtimes G$ is a factor.

Krieger showed that in the case of a Krieger factor $M = L^\infty(\Omega, \mu) \rtimes G$, the invariant $S(M)$ agrees with the so-called *asymptotic ratio set* of the group G of non-singular transformations, which is computable purely in terms of the Radon-Nikodym derivatives $\frac{d(\mu \circ T_t)}{d\mu}$. Using this ratio set description, it is not hard to see that the Krieger factor M given by the infinite product Ω:

- is a factor of type III_λ if $\nu_n = 2$ and $\mu_n\{0\} = \frac{\lambda}{1+\lambda}$ for all n;
- is a factor of type III_1 if $\nu_n = 2$ and $\mu_{2n}\{0\} = \frac{\lambda}{1+\lambda}, \mu_{2n+1}\{0\} = \frac{\kappa}{1+\kappa}$, for all n, *provided* that $\{\lambda, \kappa\}$ generates a dense multiplicative subgroup of $\mathbb{R}_+^\times$;
- can be of type III_0. □

Among all factors, Connes identified one tractable class - the so-called injective factors - which are ubiquitous and amenable to classification. To start with, he established the equivalence of several (seemingly quite disparate) requirements on a von Neumann algebra $M \subset \mathcal{L}(\mathcal{H})$ - ranging from injectivity (meaning the existence of a projection of norm one from $\mathcal{L}(\mathcal{H})$ onto M) to 'approximate finite dimensionality' (meaning $M = (\cup_n A_n)''$ for some increasing sequence $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots$ of finite-dimensional $*$-subalgebras). In the same paper ([Con2]), Connes essentially finished the complete classification of injective factors. Only the injective III_1 factor withstood his onslaught; but eventually even it had to surrender to the technical virtuosity of Haagerup ([Haa]) a few years later!

In the language we have developed thus far, the classification of injective factors may be summarized as follows:

- every injective factor is isomorphic to a Krieger factor;
- up to isomorphism, there is a unique injective factor of each type with the solitary exception of III_0;
- injective factors of type III_0 are classified (up to isomorphism) by an invariant of an ergodic theoretic nature called the *flow of weights*; unfortunately, coming up with a crisp description of this invariant, which is simultaneously accessible to the non-expert and is consistent with the stipulated size of this survey, is beyond the scope of this author.

The interested reader is invited to browse through one of the books ([Con3],[Sun] and [Dix]) for further details; the third book is the oldest (a classic but the language has changed a bit since it was written), the second is more recent (but quite sketchy in many places), and the first is clearly the best choice (if one has the time to read it carefully and digest it). Alternatively, the interested reader might want to browse through the encyclopediac treatments [KadRin] or [Tak1].

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