

A characterization of freeness by invariance under quantum spreading

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Definition

A sequence (X_1, \dots, X_n) of random variables is called

- *Exchangeable* if their joint distribution is invariant under permutations:

$$(X_1, \dots, X_n) \sim_d (X_{\omega(1)}, \dots, X_{\omega(n)}), \quad (\omega \in S_n).$$

- *Spreadable* if their joint distribution is invariant under taking subsequences:

$$(X_1, \dots, X_k) \sim_d (X_{l_1}, \dots, X_{l_k}), \quad (1 \leq l_1 < \dots < l_k \leq n).$$

- Note that independent and identically distributed (i.i.d.) \Rightarrow exchangeable \Rightarrow spreadable.
- Reverse implications fail for finite sequences.

Theorem (de Finetti '30s)

Any infinite exchangeable sequence of random variables is conditionally i.i.d.

Theorem (Ryll-Nardzewski '57)

Any infinite spreadable sequence of random variables is conditionally i.i.d.

- So for infinite sequences, the conditions of being exchangeable, spreadable and conditionally i.i.d. are all equivalent.
- For finite exchangeable sequences, Diaconis and Freedman have obtained approximate de Finetti type results.

Noncommutative context

- We will consider sequences (x_1, \dots, x_n) of (self-adjoint) elements of a von Neumann algebra M with a fixed tracial state τ .
- The *joint distribution* of such a sequence is the collection of *noncommutative joint moments*

$$\tau(x_{i_1} \cdots x_{i_k}), \quad (k \in \mathbb{N}, 1 \leq i_1, \dots, i_k \leq n)$$

- Alternative notions of “independence” in this context, most notably Voiculescu’s *free independence*.
- Exchangeability is no longer strong enough to single out one type of independence. In particular freely independent and identically distributed (f.i.d.) sequences are exchangeable.
- Spreadability also no longer implies exchangeability in this context (Köstler '10).
- More on spreadability and exchangeability in the noncommutative context in the talks of Köstler and Gohm tomorrow.

Quantum exchangeability and the free de Finetti theorem

- S_n is characterized as the automorphism group of $\{1, \dots, n\}$.
- Wang ('98) showed that $\{1, \dots, n\}$ also has a “quantum” automorphism group, denoted $A_s(n)$.

Definition (Köstler-Speicher '08)

A sequence $(x_i)_{i \in \mathbb{N}}$ is *quantum exchangeable* if for each $n \in \mathbb{N}$ the joint distribution of (x_1, \dots, x_n) is invariant under “quantum permutations” coming from $A_s(n)$.

Theorem (Köstler-Speicher '08)

An infinite sequence $(x_i)_{i \in \mathbb{N}}$ in (M, τ) is quantum exchangeable if and only if it is free and identically distributed with respect to a conditional expectation.

- Fails for finite quantum exchangeable sequences (Köstler-Speicher), but still holds in an approximate sense (C. '08).

A free analogue of Ryll-Nardzewski's theorem

The goals in this talk:

- Introduce “quantum increasing sequence spaces” $A_i(k, n)$.
- Develop an appropriate notion of “quantum spreadability”.
- Sketch the following free Ryll-Nardzewski theorem:

Theorem (Köstler-Speicher '08, C. '10)

For an infinite sequence $(x_i)_{i \in \mathbb{N}}$ in (M, τ) , the following are equivalent:

- 1 $(x_i)_{i \in \mathbb{N}}$ is quantum exchangeable.
- 2 $(x_i)_{i \in \mathbb{N}}$ is quantum spreadable.
- 3 $(x_i)_{i \in \mathbb{N}}$ is free and identically distributed with respect to a conditional expectation.

Compact quantum groups (of Kac type)

Let G be a compact group.

- $C(G)$ captures the topological structure of G (Gelfand duality).
- The group structure of G can be encoded by the morphisms:

$$\begin{aligned}\Delta : C(G) &\rightarrow C(G) \otimes C(G), & \Delta(f)(x, y) &= f(xy) \\ \epsilon : C(G) &\rightarrow \mathbb{C}, & \epsilon(f) &= f(e_G) \\ S : C(G) &\rightarrow C(G)^{op}, & S(f)(x) &= f(x^{-1}).\end{aligned}$$

- A C^* -Hopf algebra is a unital C^* -algebra A together with morphisms $\Delta : A \rightarrow A \otimes A$, $\epsilon : A \rightarrow \mathbb{C}$ and $S : A \rightarrow A^{op}$ (satisfying various compatibilities).
- Heuristic formula: “ $A = C(G)$ ” where G is a *compact quantum group*.

- View S_n as consisting of permutation matrices. $C(S_n)$ is generated by the coordinate functions f_{ij} , which satisfy the relations
 - f_{ij} is a self-adjoint projection.
 - For $1 \leq i, j \leq n$,

$$\sum_{k=1}^n f_{ik} = 1 = \sum_{k=1}^n f_{kj}.$$

The Hopf algebra structure is determined by

$$\Delta(f_{ij}) = \sum f_{ik} \otimes f_{kj}$$

$$\epsilon(f_{ij}) = \delta_{ij}$$

$$S(f_{ij}) = f_{ji}.$$

- $A_s(n)$ is defined to be the universal C^* -algebra generated by “coordinates” $\{u_{ij} : 1 \leq i, j \leq n\}$ satisfying the relations above.
- $A_s(n)$ is a C^* -Hopf algebra with morphisms defined by the above equations. We sometimes use the notation “ $A_s(n) = C(S_n^+)$ ”, where S_n^+ is the *free permutation group*.

Quantum increasing sequence spaces

- For $1 \leq k \leq n$, let $I_{kn} = \{(l_1, \dots, l_k) : 1 \leq l_1 < \dots < l_k \leq n\}$. View as $n \times k$ matrices with (i, j) entry δ_{il_j} .
- $C(I_{kn})$ is generated by coordinates $\{f_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$ which satisfy
 - f_{ij} is a self-adjoint projection.
 - For $1 \leq j \leq k$,

$$\sum_{i=1}^n f_{ij} = 1.$$

- If $j < m$ and $i \geq l$ then $f_{ij}f_{lm} = 0$.
- Define $A_i(k, n)$ to be the universal C^* -algebra generated by $\{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$ satisfying the above relations.
- $A_i(k, n)$ can be viewed as a “quantum family of maps” from $\{1, \dots, k\}$ to $\{1, \dots, n\}$, in the sense of (Sołtan '09).

Quantum exchangeability

- Define $\alpha_n : \mathbb{C}\langle t_1, \dots, t_n \rangle \rightarrow \mathbb{C}\langle t_1, \dots, t_n \rangle \otimes A_s(n)$ to be the unital homomorphism determined by

$$\alpha_n(t_j) = \sum_{i=1}^n t_i \otimes u_{ij}.$$

α_n is a *coaction* of $A_s(n)$, which can be thought of as “quantum permuting” t_1, \dots, t_n .

Definition

(x_1, \dots, x_n) is called *quantum exchangeable* if

$$(\tau \otimes \text{id})\alpha_n(p(x)) = \tau(p(x)) \cdot 1_{A_s(n)}, \quad (p \in \mathbb{C}\langle t_1, \dots, t_n \rangle).$$

Explicitly,

$$\sum_{1 \leq i_1, \dots, i_k \leq n} \tau(x_{i_1} \cdots x_{i_k}) u_{i_1 j_1} \cdots u_{i_k j_k} = \tau(x_{j_1} \cdots x_{j_k}) \cdot 1_{A_s(n)}$$

Quantum spreadability

- For $1 \leq k \leq n$ define $\beta_{k,n} : \mathbb{C}\langle t_1, \dots, t_k \rangle \rightarrow \mathbb{C}\langle t_1, \dots, t_n \rangle \otimes A_i(k, n)$ to be the unital homomorphism determined by

$$\beta_{k,n}(t_j) = \sum_{i=1}^n t_i \otimes u_{ij}.$$

Definition

(x_1, \dots, x_n) is called *quantum spreadable* if

$$(\tau \otimes \text{id})\beta_{k,n}(p(x)) = \tau(p(x)) \cdot 1_{A_i(k,n)}, \quad (1 \leq k \leq n, p \in \mathbb{C}\langle t_1, \dots, t_k \rangle).$$

Explicitly,

$$\sum_{1 \leq i_1, \dots, i_m \leq n} \tau(x_{i_1} \cdots x_{i_m}) u_{i_1 j_1} \cdots u_{i_m j_m} = \tau(x_{j_1} \cdots x_{j_m}) \cdot 1_{A_i(k,n)}$$

for $1 \leq k \leq n$ and $1 \leq j_1, \dots, j_m \leq k$

Quantum exchangeability implies quantum spreadability

- Spreadability is clearly weaker than exchangeability, as it requires invariance under fewer transformations.
- Indeed any increasing sequence $1 \leq l_1 < \dots < l_k \leq n$ can be extended to a permutation $\omega \in S_n$ with $\omega(j) = l_j$ for $1 \leq j \leq k$.
- ω is unique if one requires additionally that $\omega(k+1) < \dots < \omega(n)$.
- For example:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- Dualizing $I_{kn} \hookrightarrow S_n$ gives $C(S_n) \twoheadrightarrow C(I_{k,n})$.

Extending quantum increasing sequences to quantum permutations

Proposition

Let $\{u_{ij} : 1 \leq i, j \leq n\}$ and $\{v_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$ be the coordinates on $A_s(n)$ and $A_i(k, n)$, respectively. There is a unital $*$ -homomorphism $A_s(n) \rightarrow A_i(k, n)$ which sends u_{ij} to v_{ij} for $1 \leq i \leq n, 1 \leq j \leq k$.

- Idea of proof: Compute $C(S_n) \rightarrow C(I_{kn})$ in terms of their coordinates, and check that this formula still works in the quantum case.
- At $k = 2, n = 4$ the map is given by

$$(u_{ij}) \rightarrow \begin{pmatrix} v_{11} & 0 & 1 - v_{11} & 0 \\ v_{21} & v_{22} & v_{11} - v_{22} & 1 - (v_{11} + v_{21}) \\ v_{31} & v_{32} & v_{22} & (v_{11} + v_{21}) - (v_{22} + v_{32}) \\ 0 & v_{42} & 0 & v_{22} + v_{32} \end{pmatrix}$$

Moment-cumulant formula for f.i.d. sequences

- $NC(k)$ is the collection of *noncrossing* partitions of $\{1, \dots, k\}$.
- Given $\sigma \in NC(k)$ and indices j_1, \dots, j_k , define

$$\delta_\sigma(\mathbf{j}) = \begin{cases} 1, & s \sim_\sigma t \Rightarrow j_s = j_t \\ 0, & \text{otherwise} \end{cases}.$$

Theorem (Speicher)

A sequence (x_1, \dots, x_n) in (M, τ) is f.i.d. if and only if

$$\tau(x_{j_1} \cdots x_{j_k}) = \sum_{\sigma \in NC(k)} \delta_\sigma(\mathbf{j}) \sum_{\substack{\pi \in NC(k) \\ \pi \leq \sigma}} \mu(\pi, \sigma) \prod_{V \in \pi} \tau(x_1^{|V|}),$$

where μ is the Möbius function on $NC(k)$.

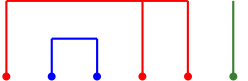
- Classical i.i.d. sequences are characterized by the same formula with $NC(k)$ replaced by $P(k)$, the set of all partitions of $\{1, \dots, k\}$.

Representation theory of S_n^+

- For $\pi \in NC(k)$ define $\rho_\pi \in (\mathbb{C}^n)^{\otimes k}$ by

$$\rho_\pi = \sum_{1 \leq j_1, \dots, j_k \leq n} \delta_\pi(\mathbf{j}) e_{j_1} \otimes \cdots \otimes e_{j_k}.$$

Example:

$\pi =$  , $\rho_\pi = \sum_{i,j,k} e_i \otimes e_j \otimes e_j \otimes e_i \otimes e_i \otimes e_k$

Theorem (Banica)

Let $u : \mathbb{C}^n \rightarrow \mathbb{C}^n \otimes A_s(n)$ be the fundamental corepresentation. Then

$$\text{Fix}(u^{\otimes k}) = \text{span}\{\rho_\pi : \pi \in NC(k)\}.$$

- For S_n the theorem holds with $NC(k)$ replaced by $P(k)$, more in Roland's talk this afternoon.

F.i.d. sequences are quantum exchangeable

- Follows from definitions that (x_1, \dots, x_n) is quantum exchangeable if and only if

$$\sum_{1 \leq i_1, \dots, i_k \leq n} \tau(x_{i_1} \cdots x_{i_k}) \cdot (e_{i_1} \otimes \cdots \otimes e_{i_k}) \in \text{Fix}(u^{\otimes k})$$

for each $k \in \mathbb{N}$.

- Moment-cumulant formula for f.i.d. sequences:

$$\begin{aligned} & \sum_{1 \leq i_1, \dots, i_k \leq n} \tau(x_{i_1} \cdots x_{i_k}) \cdot (e_{i_1} \otimes \cdots \otimes e_{i_k}) \\ &= \sum_{\sigma \in NC(k)} \left(\sum_{\substack{\pi \in NC(k) \\ \pi \leq \sigma}} \mu(\pi, \sigma) \prod_{V \in \pi} \tau(x_1^{|V|}) \right) \cdot p_\sigma \in \text{Fix}(u^{\otimes k}). \end{aligned}$$

Weingarten formula

- Define *Gram matrix*

$$G_{kn}(\pi, \sigma) = \langle p_\pi, p_\sigma \rangle = n^{\# \text{ of blocks of } \pi \vee \sigma}, \quad (\pi, \sigma \in NC(k)).$$

- G_{kn} is invertible for $n \geq 4$, let W_{kn} denote it's inverse

Theorem (Banica-Collins)

For $1 \leq i_1, j_1, \dots, i_k, j_k \leq n$,

$$\int_{S_n^+} u_{i_1 j_1} \cdots u_{i_k j_k} = \sum_{\sigma, \pi \in NC(k)} \delta_\pi(\mathbf{i}) \delta_\sigma(\mathbf{j}) W_{kn}(\pi, \sigma),$$

where \int denotes the Haar state on $A_S(n) = C(S_n^+)$.

- Difficulty is in understanding the entries of W_{kn} .
- Partial description of leading order as $n \rightarrow \infty$ given by Möbius function on $NC(k)$ (C. '08).

Theorem

For $1 \leq i_1, j_1, \dots, i_k, j_k \leq n$,

$$\int_{S_n^+} u_{i_1 j_1} \cdots u_{i_k j_k} = \sum_{\sigma \in NC(k)} \delta_{\sigma}(\mathbf{j}) \sum_{\pi \in NC(k)} \delta_{\pi}(\mathbf{i}) n^{-|\pi|} (\mu(\pi, \sigma) + O(n^{-1})).$$

Sketch of free de Finetti

Let $(x_i)_{i \in \mathbb{N}}$ be a quantum exchangeable sequence in (M, τ) .

- Define *tail algebra*

$$B = \bigcap_{n \geq 1} W^*(x_n, x_{n+1}, \dots).$$

For simplicity we will assume $B = \mathbb{C}$, so expectation onto B is τ .

- Ergodic theorem:

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{i=1}^n x_i^k = \tau(x_1^k)$$

with convergence in the strong topology. By induction on number of blocks of $\pi \in NC(k)$:

$$\lim_{n \rightarrow \infty} n^{-|\pi|} \sum_{1 \leq i_1, \dots, i_k \leq n} \delta_\pi(\mathbf{i}) x_{i_1} \cdots x_{i_k} = \prod_{V \in \pi} \tau(x_1^{|V|})$$

with convergence in the strong topology.

Sketch of free de Finetti

Let $(x_i)_{i \in \mathbb{N}}$ be a quantum exchangeable sequence in (M, τ) .

- For $1 \leq j_1, \dots, j_k \leq n$ we have

$$\tau(x_{j_1} \cdots x_{j_k}) = \sum_{1 \leq i_1, \dots, i_k \leq n} \tau(x_{i_1} \cdots x_{i_k}) \int_{S_n^+} u_{i_1 j_1} \cdots u_{i_k j_k}.$$

- Apply Weingarten formula and rearrange terms:

$$\sum_{\sigma \in NC(k)} \delta_\sigma(\mathbf{j}) \sum_{\pi \in NC(k)} W_{kn}(\pi, \sigma) \sum_{1 \leq i_1, \dots, i_k \leq n} \delta_\pi(\mathbf{i}) \tau(x_{i_1} \cdots x_{i_k}).$$

- Apply Weingarten asymptotics and take $n \rightarrow \infty$:

$$\begin{aligned} \sum_{\sigma \in NC(k)} \delta_\sigma(\mathbf{j}) \sum_{\pi \in NC(k)} (\mu(\pi, \sigma) + O(n^{-1})) n^{-|\pi|} \sum_{1 \leq i_1, \dots, i_k \leq n} \delta_\pi(\mathbf{i}) \tau(x_{i_1} \cdots x_{i_k}) \\ \xrightarrow{n \rightarrow \infty} \sum_{\sigma \in NC(k)} \delta_\sigma(\mathbf{j}) \sum_{\substack{\pi \in NC(k) \\ \pi \leq \sigma}} \mu(\pi, \sigma) \prod_{V \in \pi} \tau(x_1^{|V|}). \end{aligned}$$

Sketch of free Ryll-Nardzewski

- Use a similar “averaging” technique, what state to average against?
- Let $\{p_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$ be projections in a C^* -probability space (A, φ) such that
 - 1 The families $(\{p_{i1} : 1 \leq i \leq n\}, \dots, \{p_{ik} : 1 \leq i \leq n\})$ are freely independent.
 - 2 For $j = 1, \dots, k$, we have

$$\sum_{i=1}^n p_{ij} = 1,$$

and $\varphi(p_{ij}) = n^{-1}$ for $1 \leq i \leq n$.

- Use representation of $A_i(k, k \cdot n)$ given by

$$\begin{pmatrix} p_{11} & \cdots & p_{n1} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & p_{12} & \cdots & p_{n2} & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & p_{1k} & \cdots & p_{nk} \end{pmatrix}^T$$

State on $A_i(k, k \cdot n)$

- Representation on previous slide induces state $\psi_{k,n} : A_i(k, k \cdot n) \rightarrow \mathbb{C}$.
- $\psi_{k,n}$ is determined by

$$\begin{aligned} \psi_{k,n}(u_{(j_1-1) \cdot n + i_1, j_1} \cdots u_{(j_m-1) \cdot n + i_m, j_m}) \\ = \sum_{\sigma \in NC(m)} \delta_{\sigma}(\mathbf{j}) \sum_{\substack{\pi \in NC(m) \\ \pi \leq \sigma}} \delta_{\pi}(\mathbf{i}) n^{-|\pi|} \mu(\pi, \sigma) \end{aligned}$$

for $1 \leq j_1, \dots, j_m \leq k$ and $1 \leq i_1, \dots, i_m \leq n$, and all other expectations are zero.