'$\mathcal{O}_s$'

<u>Ex 1</u> Let Γ be a discrete countable group.

$\ell^2(\Gamma) = \{ f: \Gamma \to \mathbb{C} \mid \sum |f(\gamma)|^2 < \infty \}$ – Hilbert space.

$\delta_s : \Gamma \to \mathbb{C}$ by $\delta_s(t) = \delta_{s,t}$. Then $\{\delta_s : s \in \Gamma\}$ is orthonormal basis of $\ell^2(\Gamma)$.

For $s \in \Gamma$:- let $\lambda_s : \ell^2(\Gamma) \to \ell^2(\Gamma)$ by $\lambda_s(\delta_t) = \delta_{st}$ is unitary. Check that $\lambda_s \circ \lambda_t = \lambda_{st}$ $\forall$ s, t, Γ. So $\lambda : \Gamma \to B(\ell^2(\Gamma))$ is a unitary rep^n. (left regular rep^n)

There is also a right regular rep^n of Γ namely, $\rho : \Gamma \to B(\ell^2(\Gamma))$ by $\rho_s(\delta_t) = \delta_{ts^{-1}}$. Note λ and ρ are unitarily equivalent by the unitary $U : \ell^2(\Gamma) \to \ell^2(\Gamma)$ by $U\delta_t = \delta_{t^{-1}}$. $t \in \Gamma$.

Let $\mathbb{C}[\Gamma]$ denote the complex group alg (ring). By def^n

$\mathbb{C}[\Gamma] = \{ \sum_{s \in \Gamma} a_s s \mid$ where only finitely many $a_s \neq 0\}$. and multiplication defined by

$$\sum a_s s \cdot \sum b_t t = \sum a_s b_t \, st.$$

The group ring acquire an involution given by

$(\sum a_s s)^* = \sum \bar{a}_s s^{-1}$.

<u>Note</u> that the left regular rep^n λ can be extended to a injective $*$-hom $\mathbb{C}[\Gamma] \to B(\ell^2(\Gamma))$. which is also denoted by λ.

The reduced C^*-alg of Γ denoted by $C^*_\lambda(\Gamma)$ or $C^*_r(\Gamma)$ is the completion in norm of $\mathbb{C}[\Gamma]$ where $\|x\|_r = \|\lambda(x)\|_{B(\ell^2)}$

though isomorphic to $C^*_\lambda(\Gamma)$ it is sometimes fruitful to consider $C^*_\rho(\Gamma)$ which is the closure of the right regular rep^n

Special Case $\Gamma = \mathbb{Z}$ then $C^*_\lambda(\Gamma) = C(\mathbb{T})$. Indeed, the Fourier transform $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ is a unitary and conjugates, $\mathcal{F}^{-1} \lambda_n \mathcal{F} = M_{z^n}$. on the circle.

By Stone - Wierstrass $C^*_\lambda(\Gamma) \cong C(\mathbb{T})$.

The full group C^*- alg of Γ denoted by $C^*(\Gamma)$ is the completion of $\mathbb{C}[\Gamma]$ with respect to the norm

$$\|x\|_u = \sup_\pi \|\pi(x)\|$$

where the supremum is taken over all cyclic $*$-representations π of $\mathbb{C}[\Gamma]$ on $B(H)$ (H - varying). Evidently $C^*(\Gamma)$ enjoys the following universal property :-

Prop :- Let $\pi : \Gamma \to B(H)$ be any unitary repn of Γ. Then there is a unique $*$-hom $\Phi : C^*(\Gamma) \to B(H)$ s.t $\Phi(s) = \pi(s)$.

In particular $C^*(\Gamma)$ always has a character i.e 1-dim repn coming from the trivial repn $\Gamma \to \mathbb{C} = B(\mathbb{C})$.

Note that $\tau : C^*_\lambda(\Gamma) \to \mathbb{C}$ by $\tau(x) = \langle x\delta_e, \delta_e\rangle$ is a faithful trace.

Crossed Products :-

Defn Let Γ be a countable discrete group and A unital C^*- alg. An action of Γ on A is a group hom $\alpha : \Gamma \to \mathrm{Aut}(A)$ ($*$-automorphisms). A C^*- alg. equipped with a Γ action is called a Γ - C^*-algebra.

Goal :- is to construct a C^*-alg. that encodes the action of Γ on A (which in group theory is called semi-direct products).

'α_s'

Denote by $C_c(\Gamma, A) = \{ f : \Gamma \to A : f \text{ is finitely supported} \}$ is a linear space. A typical element in $C_c(\Gamma, A)$ is written as a finite sum $\sum_{s \in \Gamma} a_s s$, $a_s \in A$.

Equipp $C_c(\Gamma, A)$ with a α- twisted convolution product and $*$- operations as follows. For $x = \sum_{s \in \Gamma} a_s s$, $y = \sum_{t \in \Gamma} b_t t$

define
$$x *_\alpha y = \sum_{s,t \in \Gamma} a_s \alpha_s(b_t) st \quad \text{and}$$
$$x^* = \sum_{s \in \Gamma} \alpha_{s^{-1}}(a_s^*) s^{-1}.$$

This is a generalization of convolution of functions in $\ell^2(\mathbb{Z}) / \ell^1(\mathbb{Z})$. We are trying to turn $C_c(\Gamma, A)$ into a $*$- alg when the action of Γ becomes inner then the formal calculation explains the def^n.

$$\Big(\sum_{s \in \Gamma} a_s s\Big)\Big(\sum_{s \in \Gamma} b_t t\Big) = \sum_{st} a_s s b_t t = \sum_{s,t} a_s s b_t s^* st \quad (s^* = s^{-1})$$
$$= \sum a_s \alpha_s(b_t) st.$$

Def^n A covariant rep^n (u, π, H) of a Γ- C^*-alg A consists of a unitary rep^n (u, H) of Γ and a $*$ rep^n (π, H) of A s.t. $u_s \pi(x) u_s^* = \pi(\alpha_s(x))$ $\forall s, \in \Gamma$, $x \in A$.

Check that :- every co-variant rep^n like above gives rise to a $*$- rep^n of $C_c(\Gamma, A)$ (Easy).

Def^n The full crossed product, (sometimes called the universal crossed product) of a C^*- dynamical system (A, α, Γ), denoted by $A \rtimes_\alpha \Gamma$ is the completion of $C_c(\Gamma, A)$ with respect to the norm

$$\|x\|_u = \|\sup \pi(x)\|$$ when the supremum is taken over all cyclic $*$-homs $\pi : C_c(\Gamma, A) \longrightarrow B(H)$.

It is not obvious to see why $\| \ \|_u$ is a norm and why such exp^n exists, but there are lots of such π and $\| \ \|_u$ is indeed norm.

Tensor Product of Hilbert Spaces :-

Let H and K be Hilbert spaces with inner product $\langle \, , \rangle_H$, $\langle \, , \rangle_K$.

Let E, F, G be vector spaces over $\mathbb{C}$. A map $\varphi : E \times F \to G$ is bilinear if

(i) $\varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y)$.

(ii) $\varphi(\lambda x, y) = \lambda \varphi(x, y)$.

(iii) $\varphi(x, y_1 + y_2) = \varphi(x, y_1) + \varphi(x, y_2)$.

(iv) $\varphi(x, \lambda y) = \lambda \varphi(x, y)$ $\quad \lambda \in \mathbb{C}$.

Def^n A tensor product of H and K denoted by $H \otimes K$ is a Hilbert space, together with a bilinear map $\varphi : H \times K \longrightarrow H \otimes K$ s.t

(i) the set of vectors of the form $\varphi(x, y)$ $x \in H$, $y \in K$ is total in $H \otimes K$.

(ii) $\langle \varphi(x_1, y_1), \varphi(x_2, y_2) \rangle_{H \otimes K} = \langle x_1, x_2 \rangle_H \langle y_1, y_2 \rangle_K$.

It is customary to write $\varphi(x, y) = x \otimes y$.

Thus the tensor product of H and K is a Hilbert space $H \otimes K$ and a map $(x, y) \longmapsto x \otimes y$ s.t

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$$

$$(\lambda x) \otimes y = \lambda(x \otimes y).$$

$$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$$

$$x \otimes \lambda y = \lambda (x \otimes y)$$ plus (i) and (ii).

Note that
$$\| x \otimes y \| = \| x\|_H \; \|y\|_K .$$

Sometimes it is convenient to think of $H \otimes K \cong \bigoplus_{\dim(K)} H$.

or ($\cong \bigoplus_{\dim H} K$). ($W : \bigoplus_{\dim K} H \to H \otimes K$

by $W (\sum \oplus x_b) \mapsto \sum x_b \otimes y_b$ $\{y_b\}$ onb of K.

Reduced Crossed product :-

Begin with a faithful rep^n π of $A \subseteq B(H)$.

Define a new rep^n of A on $H \otimes \ell^2(\Gamma)$ by
$$\pi(x) (h \otimes \delta_g) = \alpha_{g^{-1}}(x) (h) \otimes \delta_g . \quad \text{where}$$
δ_g is cannonical basis of $\ell^2(\Gamma)$.

The point of doing this is that now the left regular rep^n of Γ spatially implements the action α.

ie
$$(1 \otimes \lambda_s) \pi(x) (1 \otimes \lambda_s^*) (h \otimes \delta_g) \qquad g \in \Gamma . \; h \in H .$$
$$= (1 \otimes \lambda_s) \pi(x) (h \otimes \delta_{s^{-1}g})$$
$$= (1 \otimes \lambda_s) \left(\alpha_{g^{-1}s} (x) (h) \otimes \delta_{s^{-1}g} \right)$$
$$= \alpha_{g^{-1}} (\alpha_s (x)) (h) \otimes \delta_g$$
$$= \cancel{\alpha_{g^{-1}} (\alpha_s (x)) (\pi (x))} \; \pi (\alpha_s (x)) (h \otimes \delta_g)$$

Thm This is a covariant rep^n.

The reduced crossed product of A with Γ written as $A \rtimes_{\alpha, r} \Gamma$ is the closure of $C_c(\Gamma, A)$ in $B(H \otimes \ell^2(\Gamma))$.

H, K Hilbert space, $H \otimes K$ tensor product.

Given $x \in B(H)$ $\exists$ a operator $x \otimes 1$ in $B(H \otimes K)$ s.t

$(x \otimes 1)(\xi \otimes \eta) = x\xi \otimes \eta$. $x \otimes 1 \rightsquigarrow \begin{pmatrix} x & & \\ & x & \\ & & x \end{pmatrix}$ bdd.

Also given $y \in B(K)$ $\exists$ $1 \otimes y \in B(H \otimes K)$ s.t

$(1 \otimes y)(\xi \otimes \eta) = \xi \otimes y\eta$. Note y has a matrix repn. $((y_{ab}))$

so $(1 \otimes y) \rightsquigarrow (\delta_{ab} I_{B(H)})$

so given ~~x, y~~ $x \in B(H)$, $y \in B(K)$ then exists $x \otimes y \in B(H \otimes K)$ s.t $(x \otimes y)(\xi \otimes \eta) = x\xi \otimes y\eta$. and

$\|x \otimes y\| = \|x\| \|y\|$.

Note $(x \otimes y)^* = x^* \otimes y^*$, $(x \otimes y)^{-1} = x^{-1} \otimes y^{-1}$ and so on.

Let $A \subseteq B(H)$ and $B \subseteq B(K)$ be C^*-subalgebra.

Then $\{ \sum x_i \otimes y_i : x_i \in A, y_i \in B \} \subseteq B(H \otimes K)$ is a $*$-subalgebra. denoted by $A \otimes_{alg} B$. Close $A \otimes_{alg} B$ in $B(H \otimes K)$ in norm. to get a C^* alg. denoted by $A \otimes_{min} B$

C^*-norms :- A and B are C^*-algs and $A \otimes_{alg} B$ be their algebraic tensor product. A C^*-alg $\mathscr{C}$ can be reasonably regarded as a tensor product of A and B if it contains $A \otimes_{alg} B$ as a dense subalgebra. Given such a C^* alg $\mathscr{C}$ its norm $|||\ |||$ restricts to a norm on $A \otimes_{alg} B$. so that $(A \otimes_{alg} B, |||\ |||)$ is a normed alg s.t $|||s^*s|||^2 = |||s|||^2$ $\forall s \in A \otimes_{alg} B$. Moreover $\mathscr{C}$ is the completion of $(A \otimes_{alg} B, |||\ |||)$.

Given two C^*-algs then are in principle uncountably many choices C^*-norms $|||\ |||_\alpha$ one can put on $A \otimes_{alg} B$.

Each C^*-norm is a cross norm i.e $\|a \otimes b\|_\alpha = \|a\|_\alpha \|b\|_\alpha = \|a\| \|b\|$.

So in principle there are many C^*-tensor products of 2 C^*-algs. given by $\| \ \|_\alpha$ but

$$\| \ \|_{min} \leq \| \ \|_\alpha \leq \| \ \|_{max}.$$