Representations :-

Def^n A representation of a C^*-alg on a Hilbert space H is a $*$-homomorphism $\Phi : A \longrightarrow B(H)$. If in addition Φ is one-to-one it is said to be a faithful rep^n.

If Φ is a rep^n of A on H from our previous discussion $\Phi(1) = 1$ and $\|\Phi(A)\| \leq \|a\|$ $\forall$ $a \in A$. Note $\|\Phi(a)\| = \|a\|$ if Φ is faithful.

Observe $\operatorname{Ker} \Phi = \{x \in A : \Phi(x) = 0\}$ is a closed two sided ideal in A.

A rep^n Φ is said to be cyclic if $\exists$ a non zero vector $\xi \in H$ s.t $\overline{A\xi} = H$. In this case ξ is said to be a cyclic vector for Φ. A vector $0 \neq \xi \in H$ is said to be separating for Φ if $\Phi(x)\xi = 0 \Rightarrow \Phi(x) = 0$.

Let $\Phi : A \to B(H)$ rep^n and $\xi \in H$. Let $\rho : A \to \mathbb{C}$ by

$$\rho(x) = \omega_\xi(\Phi(x)) \overset{def^n}{=} \langle \Phi(x)\xi, \xi \rangle.$$

If $\|\xi\| = 1$ note that ρ is a state.

Prop Let ρ be a state on a C^*-alg A. The set

$$\mathcal{L}_\rho = \{x \in A : \rho(x^*x) = 0\}$$

is a closed left ideal in A and $\rho(y^*x) = 0$ $\forall$ $x \in \mathcal{L}_\rho$, $y \in A$. The equation

$$\langle x + \mathcal{L}_\rho, y + \mathcal{L}_\rho \rangle = \rho(y^*x)$$

defines a definite inner product $\langle\ ,\ \rangle$ on the quotient vector space $A/\mathcal{L}_\rho$.

Proof:- Since ρ is positive (hence hermitian) we can define an inner product, $\langle x, y \rangle_0 = \rho(y^*x)$ on A. and let $\mathcal{L}_\rho = \{x \in A : \langle x, x \rangle_0 = 0\}$. By Cauchy Schwarz, $\mathcal{L}_\rho$ is a subspace of A. Then the equation,

$$\langle x + \mathcal{L}_\rho, y + \mathcal{L}_\rho \rangle = \langle x, y \rangle_0 = \rho(y^*x)$$

defines unambiguously an inner product on $A/\mathcal{L}_\rho$ which is definite.

If $x \in \mathcal{L}_\rho$ and $y \in A$ then

$$|\rho(y^*x)| \leq \rho(y^*y)\,\rho(x^*x) = 0. \quad \text{So } \rho(y^*x) = 0$$

Replacing y by y^*yx it follows that

$$\rho((yx)^*yx) = \rho((y^*yx)^*x) = 0 \Rightarrow yx \in \mathcal{L}_\rho.$$

Thus $\mathcal{L}_\rho$ is a left ideal in A and is closed

ρ is cont.

<u>Thm</u> (GNS construction) (Gelfand-Naimark-Segal).

Let ρ be a state on a C^*-alg A. Then is a cyclic rep^n π_ρ of A into a Hilbert space H_ρ together with a cyclic vector Ω_ρ s.t $\rho = \omega_{\Omega_\rho} \circ \pi_\rho$ ie

$$\rho(x) = \langle \pi_\rho(x)\Omega_\rho, \Omega_\rho\rangle \quad x \in A.$$

<u>Proof</u>:- let $\mathcal{L}_\rho$ as before be the left-kernel of ρ. The quotient vector space $A/\mathcal{L}_\rho$ is a pre-Hilbert space relative to the definite inner product

$$\langle x + \mathcal{L}_\rho, y + \mathcal{L}_\rho\rangle = \rho(y^*x) \quad \forall\, y, x \in A.$$

Let H_ρ denote the completion of $(A/\mathcal{L}_\rho, \langle\ \rangle)$ ie H_ρ is a Hilbert space.

Fix $x \in A$. For $y \in A$ define

$$\pi_\rho(x)(y + \mathcal{L}_\rho) = xy + \mathcal{L}_\rho.$$

Well definedness:- let $y_1 + \mathcal{L}_\rho = y_2 + \mathcal{L}_\rho$ then $y_1 - y_2 \in \mathcal{L}_\rho$. $\Rightarrow$ $x(y_1 - y_2) \in \mathcal{L}_\rho$ as $\mathcal{L}_\rho$ is left-ideal.

$\Rightarrow$ $xy_1 + \mathcal{L}_\rho = xy_2 + \mathcal{L}_\rho$. Thus π_ρ is well defined.

Clearly, $\pi_\rho(x)$ is a linear possibly unbounded operator.

Note that $x^*x \leq \|x\|^2 \cdot 1$. Thus,

$$\|\pi_\rho(x)(y + \mathcal{L}_\rho)\|_2^2 = \langle xy + \mathcal{L}_\rho, xy + \mathcal{L}_\rho\rangle = \rho((xy)^*xy) =$$

$$= \rho(y^*x^*xy) \leq \rho(y^*\|x\|^2 y) = \|x\|^2\langle y + \mathcal{L}_\rho, y + \mathcal{L}_\rho\rangle$$

Consequently,

$$\| \pi_\rho(x)(y+\mathcal{L}_\rho) \| \le \|x\| \, \|y+\mathcal{L}_\rho\|$$

Thus $\pi_\rho(x)$ extends by density to a bounded operator on H_ρ of norm at most $\|x\|$.

i.e. $\|\pi_\rho(x)\| \le \|x\|$.

Since $\pi(1)(y+\mathcal{L}_\rho) = y + \mathcal{L}_\rho \Rightarrow \pi(1) = 1_{B(H_\rho)}$.

Now let $x_1, x_2 \in A$.

$$\therefore \quad \pi_\rho(\alpha x_1 + x_2)(y+\mathcal{L}_\rho) = (\alpha x_1 + x_2)y + \mathcal{L}_\rho$$
$$= \alpha x_1 y + \mathcal{L}_\rho + x_2 y + \mathcal{L}_\rho$$
$$= (\alpha \pi_\rho(x_1) + \pi_\rho(x_2))(y+\mathcal{L}_\rho).$$

By density again π is linear.

Again
$$\pi_\rho(x_1 x_2)(y+\mathcal{L}_\rho) = x_1 x_2 y + \mathcal{L}_\rho$$
$$= \pi_\rho(x_1)(x_2 y + \mathcal{L}_\rho)$$
$$= \pi_\rho(x_1)\pi_\rho(x_2)(y+\mathcal{L}_\rho).$$

$\Rightarrow$ π_ρ is hom.

Now
$$\langle \pi_\rho(y)(x_1+\mathcal{L}_\rho), (x_2+\mathcal{L}_\rho) \rangle = \langle y x_1 + \mathcal{L}_\rho, x_2 + \mathcal{L}_\rho \rangle$$
$$= \rho(x_2^* y x_1) = \rho((y^* x_2)^* x_1)$$
$$= \langle x + \mathcal{L}_\rho, y^* x_2 + \mathcal{L}_\rho \rangle = \langle x + \mathcal{L}_\rho, \pi_\rho(y^*)(x_2+\mathcal{L}_\rho) \rangle$$

Thus $\pi_\rho(y)^* = \pi_\rho(y^*)$.

So π_ρ is a $*$-hom.

Let $\Omega_\rho = 1 + \mathcal{L}_\rho$. Then $\langle \pi_\rho(x)\Omega_\rho, \Omega_\rho \rangle$
$$= \langle x + \mathcal{L}_\rho, 1 + \mathcal{L}_\rho \rangle$$
$$= \rho(x). \quad \text{and}$$

$\|\Omega_\rho\|^2 = 1. = \langle 1+\mathcal{L}_\rho, 1+\mathcal{L}_\rho \rangle$.

Clearly $\overline{\pi_\rho(A)\Omega_\rho} = \overline{\{ x + \mathcal{L}_\rho : x \in A \}} = H_\rho$ by def^n.

Prop^n Suppose ρ is a state of a C^*-alg A. Let π be a cyclic rep^n of A in some Hilbert space $\mathcal{H}$ with a cyclic vector ξ s.t $\rho = \omega_\xi \circ \pi$. Let $(H_\rho, \pi_\rho, \Omega_\rho)$ denote the GNS triple associated to ρ. Then there exist a unitary $U: H_\rho \to H$ s.t $\xi = U\Omega_\rho$ and $\pi(x) = U\pi_\rho(x)U^*$, $x \in A$.

Proof:- For $x \in A$,

$$\|\pi(x)\xi\|^2 = \langle \pi(x)\xi, \pi(x)\xi\rangle = \langle \pi(x^*x)\xi, \xi\rangle$$
$$= \rho(x^*x) = \langle \pi_\rho(x^*x)\Omega_\rho, \Omega_\rho\rangle = \|\pi_\rho(x)\Omega_\rho\|^2.$$

If $x, y \in A$ and, $\pi_\rho(x)\Omega_\rho = \pi_\rho(y)\Omega_\rho$ then from the above it follows that $\pi(x)\xi = \pi(y)\xi$. Thus,

$H_\rho \ni U_0 \pi_\rho(x)\Omega_\rho \longmapsto \pi(x)\xi$ ~~$\in$~~ H, $x \in A$, defines a isometry from $\pi_\rho(A)\Omega_\rho$ onto $\pi(x)\xi$. Since both rep^n are cyclic U_0 extends to an unitary (by polarization identity) from H_ρ to H. and, $U\Omega_\rho = \pi(1)\xi = \xi$.

Now
$$U\pi_\rho(x)U^*\pi(y)\xi = U\pi_\rho(x)\pi_\rho(y)\Omega_\rho$$
$$= U\pi_\rho(xy)\Omega_\rho$$
$$= \pi(xy)\xi$$
$$= \pi(x)\pi(y)\xi. \quad \forall y \in A.$$

ie $U\pi_\rho(x)U^* = \pi(x)$. $\square$.

This last equation above is said ~~to be~~ as π is unitarily equivalent to π_ρ.

Cor Let ξ be a unit vector in a Hilbert space H. and $A \subseteq B(H)$ be a C^*-subalgebra. Let $\rho: A \to \mathbb{C}$ by $\rho = \omega_\xi|_A$. The rep^n π_ρ obtained from ρ in the GNS construction is unitarily equivalent to the rep^n ~~B(H)~~ $A \ni x \longmapsto x|_{\overline{A\xi}}$. The unitary $U: H_\rho \to \overline{A\xi}$ that implements the equivalence can be chosen s.t $U\Omega_\rho = \xi$.

$\overline{A\xi}$'

Proof:- This follows from the above since $\Pi : A \xrightarrow{\quad} B(\overline{A\xi})$ by $x \to x|\overline{A\xi}$ is cyclic with cyclic vector ξ and $\rho = \omega_\xi \circ \Pi$.

Thm If x is a non-zero element of a C^*-alg A, then there is a pure state ρ s.t $\pi_\rho(x) \neq 0$.

Proof:- Have proved before that there is a pure state ρ of A s.t $\rho(x) \neq 0$. $\therefore$ $\langle \pi_\rho(x)\Omega_\rho, \Omega_\rho\rangle = \rho(x) \neq 0$.

$\therefore$ $\pi_\rho(x) \neq 0$.

Def^n A state ρ on a C^*-alg. is said to be faithful if $\rho(x) = 0$ and $x \in A_+$ $\Rightarrow$ $x = 0$.

(equivalently, $\rho(x^*x) = 0 \Rightarrow x = 0$.

Thm :- let ρ be a faithful state of a C^*-alg A. Then π_ρ is faithful. (i.e injective).

Proof:- let $x \in A$ and $\pi_\rho(x) = 0$. So

$$\rho(x^*x) = \langle \pi_\rho(x^*)\,\pi_\rho(x)\,\Omega_\rho, \Omega_\rho\rangle = 0$$

$\Rightarrow$ $x = 0$. So π_ρ is a $*$-isomorphism onto its range.

Let A be a C^*-alg and $H_b : b \in B$ be a family of Hilbert spaces and $\pi_b : A \to B(H_b)$ be rep^ns of A. Then as noted, $\|\pi_b(x)\| \leq \|x\|$ $\forall x \in A$ and $\forall b \in B$.

Let $H = \bigoplus_{b \in B} H_b$. Then define a rep^n $\bigoplus \pi_b : A \to B(H)$ by $(\bigoplus \pi_b)(x) = \bigoplus_b \pi_b(x)$. Check that this is a rep^n and $\|(\bigoplus_b \pi_b)(x)\| \leq \|x\|$. $\forall x \in A$.

Theorem (Gelfand - Naimark) Every C^*-alg. A has a faithful rep^n on some Hilbert space.

Proof:- Let $\mathcal{S}$ be the collection of states of A.

For each $\rho \in \mathcal{S}$ perform the GNS construction to get the triple, $(H_\rho, \pi_\rho, \Omega_\rho)$. Let $H = \bigoplus_{\rho \in \mathcal{S}} H_\rho$ and let $\pi = \bigoplus_{\rho \in \mathcal{S}} \pi_\rho$. Let $x \in A$ be s.t $\pi(x) = 0$. then $\pi_\rho(x) = 0 \quad \forall \rho \in \mathcal{S}$. Then for,

$\rho(x) = \langle \pi_\rho(x)\Omega_\rho, \Omega_\rho \rangle = 0 \quad \forall \rho \in \mathcal{S}. \Rightarrow x = 0.$

So π is isometric. $\square$.

According Gelfand-Naimark thm can be rephrased as if A is a C^*-alg then A is $*$-isomorphic to some C^* subalgebra of $B(H)$ for some H.

(Can reduce the size of the huge direct sum by summing over pure states).

Thm If A is a separable C^*-alg, then A admits a faithful rep^n on some separable Hilbert space.

Proof:- Since A is separable, (A_1^*, w^*) is metrizable and is a compact metric space. Thus $(\mathcal{S}(A), w^*)$ is compact metric space. Let $\{\rho_n\} \subseteq \mathcal{S}(A)$ be a w^*-dense set. **Note** as A is separable, H_ρ is separable. Consider

Thm $\bigoplus H_{\rho_n}$ is separable Hilbert space.

$\bigoplus \pi_{\rho_n}$. This rep^n is faithful as, $(\bigoplus \pi_{\rho_n})(x) = 0$

$\Rightarrow \pi_{\rho_n}(x) = 0. \Rightarrow \rho_n(x) = \langle \pi_{\rho_n}(x)\Omega_\rho, \Omega_\rho \rangle = 0.$

$\Rightarrow \rho(x) = 0 \quad \forall \rho \in \mathcal{S}(A) \Rightarrow x = 0 \quad \square.$

$\pi(x)(y + \lambda_\rho) = yx + \lambda_\rho.$

$\therefore \quad \| \pi(x)(y + \lambda_\rho) \|^2 = \langle yx + \lambda_\rho, \; yx + \lambda_\rho \rangle$

$= \rho((yx)^* yx).$

$= \rho(x^* y^* y x).$

$\nleq \|x\|^2 \; \|y + \lambda_\rho\|^2.$

Suppose ρ is a trace i.e $\rho(xy) = \rho(yx) \; \forall \; x, y.$

then

$\| \pi(x)(y + \lambda_\rho) \|^2 = \rho(x^* y^* y x)$

$= \rho(y^* y x x^*).$

$= \rho(y x x^* y^*).$

$\leq \|x^*\|^2 \; \rho(y y^*)$

$= \|x^*\|^2 \; \rho(y^* y)$

$= \|x\|^2 \; \|y + \lambda_\rho\|^2.$

i.e $\pi(x)$ is bdd operator.

Again $\pi(xy)(z + \lambda_\rho) = (z\,x)y + \lambda_\rho$

$= \cancel{\lambda_{yz}} \cdot \pi(y)(zx + \lambda_\rho)$

$= \pi(y)\pi(x)(z + \lambda_\rho).$

$\therefore \quad \pi(xy) = \pi(yx)$ i.e

π is anti-homomorphism (reverses products).

If ρ is faithful trace so π is isometric.

'$\mathfrak{H}_\tau$'

$M_2(\mathbb{C})$ with $\tau = tr(\cdot)$ $M_2(\mathbb{C})$ oririnally acts on $\mathbb{C}^2$.

$$\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle_\tau = tr\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$$

$$= tr\left(\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$$

$$= tr \begin{pmatrix} a^*a + c^*c & a^*b + c^*d \\ b^*a + d^*a & b^*b + d^*d \end{pmatrix}$$

$$= \frac{1}{2}\left(a^*a + b^*b + c^*c + d^*d \right)$$

$$= 0.$$

$$\Rightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0.$$

$\therefore$ $\mathcal{L}_\rho = 0.$ $\Rightarrow$ $M_2(\mathbb{C}) / \mathcal{L}_\rho = M_2(\mathbb{C}).$

Complete $M_2(\mathbb{C})$ with $\| \ \|_2$. (Hilbert space norm).

But $M_2(\mathbb{C})$ is finite dimensional thus closed. in any norm.

$\therefore$ $H_\tau = M_2(\mathbb{C})$.

$$\therefore \quad \pi_\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix}$$

$$\pi'_\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Check that $\pi_\tau(x) \cdot \pi'_\tau(y) = \pi'_\tau(y)\, \pi_\tau(x)$.

and also check $\pi_\tau(M_2(\mathbb{C}))' = \pi'_\tau(M_2(\mathbb{C}))$.

So the algebra and the comutant is ~~of~~ same size. While if we think $M_2(\mathbb{C})$ acting on $\mathbb{C}^2$ then $M_2(\mathbb{C})' = \mathbb{C}1$.

$M_2(\mathbb{C})$, $\qquad \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \{\ a.$

ρ is a state.

<u>Now</u> $\quad \rho\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = 0$

$\Rightarrow \qquad a^*a + c^*c = 0.$

$\Rightarrow \qquad a = 0, \quad c = 0.$

$\therefore \quad L_\rho = \left\{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} : b, d \in \mathbb{C} \right\}.$

$\therefore \quad M_2(\mathbb{C}) / L_\rho = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbb{C} \right\}.$

$\therefore \quad \left(M_2(\mathbb{C}) /_{L_\rho}, \ \| \ \|_\rho \right)$ is 2-dim.

$\therefore \quad \pi\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 & 0 \\ ax_3 + bx_4 & 0 \end{pmatrix}.$

Try to find the commutant of this repⁿ.

This repⁿ is still faithful but the state is not faithful.