Positive linear Functionals :- A unital C^*-alg. $A_{s.a}$: all self adjoints, A_+ all positives. Note A is a linear span of positives.

Let $\rho \in A^*$ be linear functional. Define the adjoint functional by $\rho^* : A \to \mathbb{C}$ by $\rho^*(a) = \overline{\rho(a^*)}$. $a \in A$.

say ρ is hermetian if $\rho = \rho^*$ i.e $\rho^*(a) = \rho(a) \iff \rho(a^*) = \overline{\rho(a)}$ $\forall\, a \in A$. By expressing A elements in A in real and imaginary parts, it follows ρ is hermetian if $\rho(h) = \rho^*(h) = \overline{\rho(h)}$ $\forall\, h \in A_{s.a}$. So ρ hermetian $\iff$ $\rho(h)$ is real for all $h \in A_{s.a}$.

Let $\rho : A \to \mathbb{C}$ be any functional. Then note

$$\rho = \rho_1 + i \rho_2 \quad \text{with } \rho_1 \text{ and } \rho_2 \text{ hermetian when}$$

$$\rho_1 = \frac{1}{2}(\rho + \rho^*) \quad \text{and} \quad \rho_2 = \frac{i}{2}(\rho^* - \rho).$$

Claim If ρ is hermitian, ~~then~~ and bounded,

$$\|\rho\| = \sup\{\rho(h) : h = h^*,\ \|h\| \le 1\}.$$

Indeed if $\varepsilon > 0$, choose $a \in A_1$ s.t $|\rho(a)| > \|\rho\| - \varepsilon$.

$\therefore$ For a suitable scalar c s.t $|c| = 1$

$$\|\rho\| - \varepsilon < |\rho(a)| = \rho(ca) = \overline{\rho(ca)} = \rho((ca)^*).$$

Let $h_0 = \mathrm{Re}\,(ca)$ then $\|h_0\| \le 1$ and $\|\rho\| - \varepsilon < \rho(h_0)$.

Thus $\|\rho\| \le \sup\{\rho(h) : h = h^*,\ \|h\| \le 1\}$. The other inequality is obvious.

Def^n A linear functional ρ on A is said to be positive if $\rho(a) \ge 0$ $\forall\, a \in A_+$. If further, $\rho(1) = 1$, ρ is said to be a state.

A positive linear functional is hermitian, as if $a = a^*$ then $\rho(\|a\| \cdot 1 \pm a) \ge 0$ and $\|a\| \cdot 1 \pm a \in A_+$ and

$$\rho(a) = \frac{1}{2}\Big(\rho(\|a\| \cdot 1 + a) - \rho(\|a\| \cdot 1 - a)\Big) \in \mathbb{R}.$$

Prop: f positive linear functional of A. Then

$$|f(b^*a)|^2 \le f(a^*a)\, f(b^*b). \quad a, b \in A.$$

Proof $a \in A, \Rightarrow a^*a \in A_+, \Rightarrow f(a^*a) \geq 0$. From this and since f is hermitian,

$$\langle a, b\rangle = f(b^*a) \quad a, b \in A$$

defines a inner product on A. (need not be definite). By Cauchy Schwarz,

$$|\langle a, b\rangle|^2 \le \langle a, a\rangle \langle b, b\rangle. \text{ ie}$$

$$|f(b^*a)|^2 \le f(a^*a)\, f(b^*b).$$

~~Thm Let $f \in A^*$. Then f is positive ($\Rightarrow$)~~

~~Thm :- Let $f: A \to \mathbb{C}$ be a positive linear functional.~~

~~Thm f is a~~

Thm Let $f: A \to \mathbb{C}$ be a linear functional. Then f is positive $(\Leftrightarrow)$ f is bounded and $\|f\| = f(1)$.

Proof:- Suppose f is positive. Let $a \in A$. Choose $c \in \mathbb{C}$, $|c| = 1$ s.t $cf(a) \geq 0$. ie $f(ca) \geq 0$. Let $h = \mathrm{Re}(ca)$ then $\|h\| \le \|a\|$. So $h \le \|h\|\cdot 1 \le \|a\|\cdot 1 \Rightarrow \|a\| f(1) - f(h) = f(\|a\|\cdot 1 - h) \geq 0$

$$\therefore \quad |f(a)| = f(ca) = \overline{f(ca)} = f(\bar{c}a^*) \quad \text{(hermitian)}$$

$$= f\left(\tfrac{1}{2} ca + \tfrac{1}{2}\bar{c}a^*\right)$$

$$= f(h) \le f(1)\|a\|.$$

$\Rightarrow \|f\| \le f(1) \Rightarrow \|f\| = f(1)$.

Conversely assume $\|f\| = f(1) = 1$ (else scale). Let $x \in A_+$ and let $f(x) = a + ib$. $a, b \in \mathbb{R}$. To show $a \geq 0$ $b = 0$.

Since $\sigma(x) \subseteq \mathbb{R}_+$ so for small $s > 0$, $\sigma(1 - sx) = \{1 - st : t \in \sigma(x)\} \subseteq [0, 1]$.

So $\|1 - sx\| = r(1 - sx) \le 1$. Hence $1 - sa \le |1 - s(a + ib)| = |f(1 - sx)| \le 1. \Rightarrow a \geq 0$.

Let $b_n = x - a1 + inb1$, $n \in \mathbb{N}$. Then

$\|b_n\|^2 = \|b_n^* b_n\| = \|(x - a1)^2 + n^2b^2 1\| \le \|x - a1\|^2 + n^2b^2$. Hence

$(n^2 + 2n + 1)b^2 = |f(b_n)|^2 \le \|x - a1\|^2 + n^2b^2 \quad \forall n \Rightarrow b = 0$.

So let $S(A) = \{ f : A \to \mathbb{C}$: linear bdd and $\underset{\|f\|}{f(1)} \underset{=}{=} 1\}$ be the collection of states space of A.

Then $S(A)$ is compact in the w^*_top i.e a cpt Hausdorff space.

Prop let $1, x \in A$. and $a \in \sigma(x)$. Then there exist a state f of A s.t $f(x) = a$.

Proof:- For all complex nos b, c, we have $ab + c \in \sigma(bx + c)$ and therefore, $|ab + c| \leq \|bx + c\|$. Accordingly the association $f_0(bx + c) = ab + c$ defines unambiguously a linear functional f_0 on the ~~the~~ subspace $\{bx + c : b, c \in \mathbb{C}\}$. and $f_0(x) = a$, $f_0(1) = 1$, and $\|f_0\| = 1$. By Hahn-Banach thm extend f_0 to a bdd linear functional on A s.t $\|f\| = 1. = f(1)$. By the prev thm, $f \in S(A)$.

Thm :- let $1 \in A$. and $x \in A$. Then,

(i) If $f(x) = 0 \quad \forall f \in S(A)$ then ~~f~~ $x = 0$.

(ii) If $f(x)$ is real $\forall f \in S(A)$ then $x = x^*$.

(iii) If $f(x) \geq 0 \quad \forall f \in S(A)$ then $x \geq 0$.

(iv) If x is normal then $\exists f \in S(A)$ s.t $|f(x)| = \|x\|$.

Proof:- (i) First assume $x = x^*$. $\therefore f(x) = 0 \quad \forall f \in S(A) \Rightarrow \sigma(x) = \{0\}$

$\Rightarrow \|x\| = r(x) = 0. \Rightarrow x = 0$. Next let $x = h + ik$ (Cartesian decomposition)

then $f(h) = 0. = f(k) \quad \forall f$ as $f(h), f(k)$ are real. $\therefore h = k = 0$.

(ii) $f(x)$ real $\forall f \in S(A)$. $\therefore f(x - x^*) = f(x) - f(x^*) = f(x) - \overline{f(x)}$ (hermitian) $= 0$.

$\therefore x = x^*$ by (i).

(iii) $f(x) \geq 0 \quad \forall f \in S(A) \Rightarrow x = x^*$ by (ii) and $\sigma(x) \subseteq \mathbb{R}_+$ by lemma. $\Rightarrow x \geq 0$.

(iv) x normal. So $r(x) = \|x\|$. So $\sigma(x)$ contains a scalar c ~~with~~ ~~$|c|=$~~ s.t $|c| = \|x\|$. By Prop $\exists f \in S(A)$ s.t $f(x) = c$

$\therefore |f(x)| = \|x\|$.

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<u>Thm</u> let A be a C^*-alg with 1. and f be a bounded hermitian linear functional. Then $\exists$ positive linear functionals f_+, f_- on A s.t $f = f_+ - f_-$ and $\|f\| = \|f_+\| + \|f_-\|$. These conditions determine f_+, f_- uniquely.

Proof:- Skipped.

<u>Cor</u> Every bounded linear ~~combi~~ functional on a C^* alg with 1 is a linear sum of at most 4 states.

<u>Note</u> $S(A)$ is compact and convex in the (locally convex) w^* topology. So by Krien - Milimann, $S(A)$ is the w^*-closed convex hull of its extreme pts. These extreme pts are called pure states. ~~defon~~ denoted by $P(A)$.

<u>Thus</u> we have the following:-

<u>Thm</u> let $x, 1 \in A$:

(i) If $f(x) = 0 \quad \forall f \in P(A) \Rightarrow x = 0$.

(ii) If $f(x)$ is real $\forall f \in P(A) \Rightarrow x = x^*$.

(iii) If $f(x) \geq 0 \quad \forall f \in P(A) \Rightarrow x \geq 0$.

(iv) If $xx^* = xx^*$ then $\exists$ a pure state $f_0 \in P(A)$ s.t $|f_0(x)| = \|x\|$.

<u>Proof</u> Only need to show (iv).

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By (iv) of previous Thm $\exists$ a state τ of A and a scalar c s.t. $\tau(x) = c$, $|c| = \|x\|$. let a be the complex no with $|a| = 1$ s.t. $\tau(ax) = |c| = \|x\|$.

Then by separation, $\exists f_0 \in P(A)$

$$\|x\| \geq |f_0(x)| \geq \mathrm{Re}\, \widehat{ax}(f_0) \geq \sup\{\mathrm{Re}\, \widehat{ax}(f) : f \in S(A)\}$$

$$\geq \mathrm{Re}\, \widehat{ax}(\tau) = \mathrm{Re}\, \tau(ax) = \|x\|.$$