'algs'

Def^n A $*$-homomorphism between two C^*-algs A and B is a linear map $\Phi : A \to B$ s.t $\Phi(xy) = \Phi(x)\Phi(y)$ and $\Phi(x^*) = \Phi(x)^*$ and $\Phi(1) = 1$.

Thm:- let A, B an C^*-algs and $\Phi : A \to B$ is a $*$ hom. then

(i) For $a \in A$, $\sigma(\Phi(a)) \subseteq \sigma(a)$ and $\|\Phi(a)\| \leq \|a\|$, so Φ is continuous.

(ii) If $a = a^* \in A$, $f \in C(\sigma(a))$, then $\Phi(f(a)) = f(\Phi(a))$.

(iii) If Φ is a $\underset{\text{injective}}{\underline{*\text{-isomorphism}}}$, then $\|\Phi(a)\| = \|a\|$ and $\sigma(\Phi(a)) = \sigma(a)$ $\forall$ $a \in A$. and $\Phi(A)$ is a C^*-sub alg of B.

Proof:- (i) let $\lambda \notin \sigma(a)$ $\Rightarrow$ $a - \lambda$ has an inverse b in A. Since $\Phi(1) = 1$ so $\Phi(a) - \lambda 1$ has the inverse $\Phi(b)$ in B. So $\lambda \notin \sigma(\Phi(a))$; hence $\sigma(\Phi(a)) \subseteq \sigma(a)$.

Note that $\|a\|^2 = \|a^*a\| = r(a^*a)$ then

$\|\Phi(a)\|^2 = \|\Phi(a)^*\Phi(a)\| = \|\Phi(a^*a)\| = r(\Phi(a^*a))$.

Since $\sigma(\Phi(a^*a)) \subseteq \sigma(a^*a)$ $\Rightarrow$ $r(\Phi(a^*a)) \leq r(a^*a)$.

$\Rightarrow$ $\|\Phi(a)\| \leq \|a\|$.

(ii) Now $\sigma(a)$ is larger than $\sigma(\Phi(a))$. let p_n be a sequence of polynomials s.t $p_n \to f$ uniformly on $\sigma(a)$.

$\therefore$ $p_n(a) \to f(a)$ $\Rightarrow$ $\Phi(p_n(a)) \to \Phi(f(a))$. Again $p_n(\Phi(a)) \to f(\Phi(a))$. But $p_n\Phi(a) = \Phi(p_n(a))$ $\forall n$ (as Φ is hom). So $f(\Phi(a)) = \Phi(f(a))$.

(iii) let Φ be injective, and $b = b^* \in A$. We know $\sigma(\Phi(b)) \subseteq \sigma(b)$. If strict inclusion happens $\exists$ $f : \sigma(b) \to \mathbb{C}$ cont s.t $f = 0$ on $\sigma(\Phi(b))$, $f \neq 0$. $\therefore$ $f(b) \neq 0$ but $\Phi(f(b)) = f(\Phi(b)) = 0$. contrary to injectivity. So $\sigma(\Phi(b)) = \sigma(b)$ and $r(\Phi(b)) = r(b)$.

'$\overline{n}$'

Thm if $b = a^* a$ with $a \in A$, then

$$\|a\|^2 = r(a^* a) = r(\Phi(a^* a)) = \|\Phi(a^* a)\| = \|\Phi(a)\|^2$$

ie $\|\Phi(a)\| = \|a\|$. (That $\Phi(A)$ is a C^*-sub alg is clear).

Cor let A be a C^*-alg. with $\| \; \|$. ~~Then~~ Let $\| \; \|_n$ be any new norm on A. with respect to which A is a C^*-alg. Then $\| \; \| = \| \; \|_n$.

Pf $i : (A, \| \; \|) \to (A, \| \; \|_n)$ is a $*$-hom and injective. So i is isometry.

Thm Let $\Phi : A \to B$ be a $*$-hom of two C^*-algs. Then $\Phi(A)$ is a C^*-sub alg. of B.

Proof:- $\Phi(A)$ is a unital $*$-subalg of B. To show $\Phi(A)$ is norm closed. We must prove that if $b \in B$ and $a_n \in A$ is such that $\Phi(a_n) \to b$ then $b \in \Phi(A)$. Enough to check for self adjoints a_n, b. (?) Dropping to a subsequence can assume $\|\Phi(a_{n+1}) - \Phi(a_n)\| < \frac{1}{2^n}$ $\forall n$.

let $f_n(t) = t$ on $[-\frac{1}{2^n}, \frac{1}{2^n}]$. From (ii) of prev thm $f_n = id$ on $\sigma(\Phi(a_{n+1}) - \Phi(a_n))$, and

$$\Phi(a_{n+1}) - \Phi(a_n) = f_n(\Phi(a_{n+1} - a_n)) = \Phi(f_n(a_{n+1} - a_n)) \quad (*)$$

Since $\|f_n(a_{n+1} - a_n)\| \leq \frac{1}{2^n}$ the series $a_1 + \sum_{n=1}^{\infty} f_n(a_{n+1} - a_n)$ converges to an element $a \in A$. By continuity of Φ,

$$\Phi(a) = \lim_{m \to \infty} \left(\Phi(a_1) + \sum_{n=1}^{m-1} \varphi(f_n(a_{n+1} - a_n)) \right)$$

$$= \lim_{m \to \infty} \left(\Phi(a_1) + \sum_{n=1}^{m-1} (\varphi(a_{n+1}) - \varphi(a_n)) \right)$$

$$= \lim_m \Phi(a_m)$$

$$= b. \quad \square$$

Lemma:- Let $a = a^* \in A$ and $\lambda \in \mathbb{R}$ s.t $\|a\| \leq \lambda$. Then a is positive $\iff$ $\|a - \lambda\| \leq \lambda$.

Proof:- Note $\sigma(a) \subseteq [-\lambda, \lambda]$. and

$$\|a - \lambda\| = r(a - \lambda) = \sup_{t \in \sigma(a)} |t - \lambda| = \sup_{t \in \sigma(a)} \lambda - t$$

It is apparent that $\|a - \lambda\| \leq \lambda \iff \sigma(a) \subseteq \mathbb{R}_+$.

Thm: Let A_+ denote the collection of all positives in a C^*-alg A. Then

(i) A_+ is closed in A.

(ii) $\lambda a \in A_+$ if $a \in A_+$, $\lambda \geq 0$.

(iii) $a + b \in A_+$ if $a, b \in A_+$

(iv) $ab \in A_+$ if $a, b \in A_+$ and $ab = ba$.

(v) $a \in A_+$ and $-a \in A_+$ then $a = 0$.

Proof:- (i) Note from lemma above,

$$A_+ = \{a \in A : a = a^*, \|a - \|a\|\| \leq \|a\|\}.$$

It clearly follows A_+ is closed as $*$, $\|\ \|$ are continuous operations.

(ii) $(\lambda a)^* = \lambda a^* = \lambda a$. and $\sigma(\lambda a) = \{\lambda t : t \in \sigma(a)\} \subseteq \mathbb{R}_+$

(iii) $a, b \in A_+$. From lemma $\|a - \|a\|\| \leq \|a\|$, $\|b - \|b\|\| \leq \|b\|$

Then $\|(a+b) - (\|a\| + \|b\|)\| \leq \|a\| + \|b\|$. with $\lambda = \|a\| + \|b\|$ $\geq \|a + b\|$ it follows $a + b \in A_+$.

(iv) Since $ab = ba$, so $(ab)^* = ab$. Since a, b, ab has the same spectrum in A as in the abelian C^*-alg $C^*(1, a, b)$ it follows from before

$$\sigma(ab) \subseteq \{st : s \in \sigma(a), t \in \sigma(b)\} \subseteq \mathbb{R}_+.$$

(v) $a, -a \in A_+ \Rightarrow \sigma(a) \subseteq \mathbb{R}_+ \cap (-\mathbb{R}_+) = \{0\}$.

so $r(a) = 0$. As a is self-adjoint so $\|a\| = r(a) = 0$

$\Rightarrow a = 0$. (All these properties say A_+ is a positive cone).

Prop $a = a^*$ in A. and $f \in C(\sigma(a))$. Then

(i) $f(a)$ is positive if $f(t) \geq 0 \quad \forall \; t \in \sigma(a)$.

(ii) $\|a\| \pm a \in A_+$.

(iii) $a = a_+ - a_-$ with $a_\pm \in A_+$ and $a_+ a_- = 0$

(note that $\|a\| = \max(\|a_+\|, \|a_-\|)$.

Proof:- Exercise.

Cor Every element in a C^*-alg is a linear combination of at most 4 positive.

Pf $x \in A$; $x = x_1 + i x_2$ decompose x_1, x_2.

Lemma:- $-a^*a \in A_+ \Rightarrow a = 0$.

Proof:- let $a = h + ik$ with h, k, s.a

Since $\sigma(h) \subseteq \mathbb{R}$ so $\sigma(h^2) = \{t^2 : t \in \sigma(h)\} \subseteq \mathbb{R}_+$.

then h^2 and k^2 are positives. and aa^* is self-adjoint

and $\sigma(-aa^*) \subseteq \sigma(-a^*a) \cup \{0\} \subseteq \mathbb{R}_+$.

Then $-aa^* \in A_+$.

Now $a^*a + aa^* = (h - ik)(h + ik) + (h + ik)(h - ik) = 2(h^2 + k^2)$.

$\therefore \quad a^*a = 2h^2 + 2k^2 - aa^*$. By previous lemma

a^*a is positive also, $-a^*a$ is positive. $\Rightarrow a = 0$.

Thm $a \in A$. TFAE

(i) $a \in A_+$ (ii) $a = b^2$ for some $b \in A_+$, (iii) $a = b^*b$ for some $b \in A$.

Pf:- Skipped.

Cor A C^*-alg. $a \in A_+$ and $b \in A$ then b^*ab is positive.

Pf $b^*ab = (a^{1/2}b)^* a^{1/2} b$.

Lemma:- A unital Banach alg. $\sigma(xy) \cup \{0\} = \sigma(yx) \cup \{0\}$ $x, y \in A$.

Proof:- Want to show if $\lambda \neq 0$ and $\lambda - xy \in G(A)$ then $\lambda - yx \in G(A)$

Dividing by λ can assume $\lambda = 1$.

Let $u = (1 - yx)^{-1}$. Then

$u - uyx = u - yxu = 1$. Set $v = 1 + xuy$ and note that,

$$v(1 - xy) = (1 + xuy)(1 - xy)$$
$$= 1 + xuy - xy - xuyxy.$$
$$= 1 + x(u - 1 - uyx)y$$
$$= 1. \quad \square.$$

$\boxed{\text{Or } r(ab) = r(ba)}$

Note $A_{s.a}$ is real linear closed subspace of A, so is a Banach space, with a cone of positive. So define partial order in $A_{s.a}$ by $a \leq b$ if $b - a \in A_+$. $(b - a \geq 0)$.

Check this is partial order.

Thm:- Let $a = a^*$, $b = b^* \in A$.

(i) $-b \leq a \leq b \Rightarrow \|a\| \leq \|b\|$.

(ii) $0 \leq a \leq b \Rightarrow a^{1/2} \leq b^{1/2}$.

(iii) $0 \leq a \leq b$ and a invertible $\Rightarrow b$ is invertible and $b^{-1} \leq a^{-1}$.

Proof:- First note for $x = x^*$, $-\|x\|\cdot 1 \leq x \leq \|x\|\cdot 1$

(i) $-\|b\|\cdot 1 \leq -b \leq a \leq b \leq \|b\|\cdot 1$ use functional calculus.

(ii) and (iii) Let $0 \leq a \leq b$ and a is invertible. Then by function calculus $\exists\ \lambda > 0$ s.t $a \geq \lambda\cdot 1 \Rightarrow b \geq \lambda\cdot 1$, then b is invertible.

Moreover, $0 \leq b^{-1/2} a b^{-1/2} \leq b^{-1/2} b b^{-1/2} = 1$. Thus, $\|b^{-1/2} a b^{-1/2}\| \leq 1$.

by (i).

Thus, $\|a^{+1/2} b^{-1/2}\| = \|(a^{1/2} b^{-1/2})^* a^{1/2} b^{-1/2}\|^{1/2} = \|b^{-1/2} a b^{-1/2}\|^{1/2} \leq 1$.

From this $\|a^{1/2} b^{-1} a^{1/2}\| = \|(a^{1/2} b^{-1/2})(a^{1/2} b^{-1/2})^*\| = \|a^{1/2} b^{-1/2}\|^2 \leq 1$,

whence $0 \leq a^{1/2} b^{-1} a^{1/2} \leq 1$ (Gelfand transform). $\Rightarrow b^{-1} \leq a^{-1}$.

Now, $\|b^{-1/4} a^{1/2} b^{-1/4}\| = r(b^{-1/4} a^{1/2} b^{-1/4}) = r(a^{1/2} b^{-1/2})$

$$\leq \|a^{1/2} b^{-1/2}\| \leq 1.$$

$\therefore\ 0 \leq b^{-1/4} a^{1/2} b^{-1/4} \leq 1$

$\Rightarrow a^{1/2} \leq b^{1/2}$. This proves (ii) and (iii) when a is invertible.

Given $0 \le a \le b$ $\forall\, \varepsilon > 0$ s.t.
$0 \le a + \varepsilon . 1 \le b + \varepsilon 1$ and $a + \varepsilon . 1$ is invertible.

From the previous argument,
$$(a+\varepsilon)^{1/2} \le (b+\varepsilon)^{1/2}.$$

Now let $x \in A_+$ and $f_\varepsilon : \sigma(x) \to \mathbb{R}$ be defined by $f_\varepsilon(t) = (t+\varepsilon)^{1/2}$. $\therefore f_\varepsilon(x) \in A_+$ $f_\varepsilon(x)^2 = x + \varepsilon . 1$
$\therefore f_\varepsilon(x) = (x + \varepsilon . 1)^{1/2}$. Note, $f_\varepsilon(t) \to t^{1/2}$ uniformly as $\varepsilon \to 0$. $\therefore \| (x+\varepsilon)^{1/2} - x^{1/2} \| \to 0$. As A_+ is closed taking limits $a^{1/2} \le b^{1/2}$.

<u>Prop</u> Let K be a closed left-ideal in a C^*-alg A. Then if $s \in K$ then, $s = ak$ with $a \in A$ and $k \in K \cap A_+$.

<u>Proof</u>:- First note that if $y \in K \cap A_+$ then $y^{1/2} \in K \cap A_+$. Indeed as K is left ideal so K contains all powers of y. Note $t^{1/2}$ is a uniform limit of polynomials without constant term on $\sigma(y)$. Thus $y^{1/2} \in K$ by functional calculus.

For $s \in K$ write $h = (s^*s)^{1/2}$ and $k = h^{1/2}$. Then $s^*s \in K \cap A_+$, thus $h, k \in K$. For $n = 1, 2, \ldots$ define,
$$a_n = s\left(\tfrac{1}{n} + h\right)^{-1/2} \quad \text{so that} \quad s = a_n\left(\tfrac{1}{n} + h\right)^{1/2}.$$

$$\text{Then } \|a_m - a_n\| = \left\| s\left[\left(\tfrac{1}{m} + h\right)^{-1/2} - \left(\tfrac{1}{n} + h\right)^{-1/2}\right]\right\|$$

$$= \left\|\left(\left(\tfrac{1}{m} + h\right)^{-1/2} - \left(\tfrac{1}{n} + h\right)^{-1/2}\right) s^*s \left(\left(\tfrac{1}{m} + h\right)^{-1/2} - \left(\tfrac{1}{n} + h\right)^{-1/2}\right)\right\|$$

$$= \left\|\left(\left(\tfrac{1}{m} + h\right)^{-1/2} - \left(\tfrac{1}{n} + h\right)^{-1/2}\right) h^2 \left(\left(\tfrac{1}{m} + h\right)^{-1/2} - \left(\tfrac{1}{n} + h\right)^{-1/2}\right)\right\|$$

$$= \left\|\left(f_{mn}(H)\right)^2\right\|^{1/2} = \|f_{mn}(H)\| \quad \text{when}$$

$$f_{m,n}(t) = t\left[\left(\tfrac{1}{m} + t\right)^{-1/2} - \left(\tfrac{1}{n} + t\right)^{-1/2}\right] \quad \text{on } \mathbb{R}_+.$$

Thus $\|a_m - a_n\| = \sup\{|f_{mn}(t)| : t \in \sigma(H)\}$

Note

$$\sup_{t \in \sigma(H)} |f_{m,n}(t)| = \sup_{t\in\sigma(H)} \frac{t\left|\sqrt{\frac{1}{n}+t} - \sqrt{\frac{1}{m}+t}\right|}{\sqrt{\frac{1}{n}+t}\,\sqrt{\frac{1}{m}+t}}$$

$$\leq \sup_{t\in\sigma(H)} \left|\sqrt{\frac{1}{n}+t} - \sqrt{\frac{1}{m}+t}\right| \longrightarrow 0 \text{ as } (n,m)\to\infty.$$

$\therefore$ a_n is Cauchy in A so let $\lim_n a_n = a \in A$.

$\therefore$ Passing to limits

$a = s\,h^{-1/2}$ or $s = a\,h^{1/2}$ $= aK$ $\square$.

<u>Corollary</u> Every closed two sided ideal K in a C^*-alg A is self-adjoint. (A closed two sided ideal in A)

<u>Proof</u>:- let K be a closed two sided ideal in A. let $s \in K$.

Then $s = aK = a(s^*s)^{1/4}$ when $a \in A$, $K \in K \cap A_+$. So $s^* = K a^* \in A\cdot K$. So K is

Since K is a right ideal so

self-adjoint.

~~(Next suppose that $J \subseteq K$ is closed 2)~~

<u>Def</u>^{n} A approximate identity for a Banach alg A is a net e_λ, $\lambda \in \Lambda$ which is bdd and satisfies $\lim_\lambda e_\lambda a = \lim_\lambda a e_\lambda = a$. $\forall a \in A$.

In a C^*-alg, one further demands, $0 \leq e_\lambda$, $\|e_\lambda\| \leq 1$, $e_\lambda \leq e_\mu$ when $\lambda \leq \mu$. Since Λ is directed so for each $\lambda, \mu \in \Lambda$ $\exists$ a index $\nu \in \Lambda$ s.t $e_\nu \geq e_\lambda$, $e_\nu \geq e_\mu$.

<u>Thm</u> :- Every C^*-alg has an approximate identity.

Proof:- The proof is skipped for lack of time.

'ops'

Theorem :- Let $\mathcal{J}$ be a closed both sided ideal of a C^*- alg A. Then $A/\mathcal{J}$ is a C^*- alg.

Proof :- Elements in $A/\mathcal{J}$ are denoted by $[x]$ for $x \in A$. ie $[x] = x + \mathcal{J}$. Define $[x]^* = [x^*]$. and check that this defines an involution on the Banach alg $A/\mathcal{J}$. Note that $\|[x]\| = \inf_{j \in \mathcal{J}} \|x - j\|$. Since $\mathcal{J}$ is self-adjoint and $*$ is isometric on A so $\|[x]\| = \|[x]^*\|$.

So only need to check C^*- norm.

Let e_λ be an approx. identity for $\mathcal{J}$. (Note $\mathcal{J}$ is a C^* alg).

We claim,

$$\|[x]\| = \lim_\lambda \|x - xe_\lambda\|. \quad — ①$$

Indeed $xe_\lambda \in \mathcal{J}$ so $\|x - xe_\lambda\| \geq \|[x]\|.\ \forall \lambda$ On the other hand, for $\varepsilon > 0$ $\exists$ $j \in \mathcal{J}$ s.t $\|x - j\| < \|[x]\| + \varepsilon$.

So $$\lim_{\lambda \in \Lambda} \|x - xe_\lambda\| \leq \lim_\lambda \|(x - j)(1 - e_\lambda)\| + \|j - je_\lambda\|$$

$$\leq \|x - j\| < \|[x]\| + \varepsilon.$$

Let $\varepsilon \to 0$. So ① follows.

Now $$\|[x]^*[x]\| = \lim_\lambda \|x^*x(1 - e_\lambda)\|$$

$$\geq \lim_\lambda \|(1 - e_\lambda) x^*x (1 - e_\lambda)\|$$

$$= \lim_\lambda \|a(1 - e_\lambda)\|^2 = \|[x]\|^2$$

$$= \|[x]\| \, \|[x]\|$$

$$= \|[x]^*\| \, \|[x]\|$$

$$\geq \|[x]^*[x]\|.$$

$$\Rightarrow \|[x]^*[x]\| = \|[x]\|^2. \quad \square$$