Prop Let $1, x \in B \subseteq A$ (A, B Banach algs). Then

(a) $\sigma_B(x) \supseteq \sigma_A(x)$.

(b) $\partial \sigma_B(x) \subseteq \partial \sigma_A(x)$. ($\partial$ topological boundary).

Proof:- (a) To show $\rho_B(x) \subseteq \rho_A(x)$.

Let $\lambda \in \rho_B(x) \Rightarrow x-\lambda \in \mathcal{G}(B) \Rightarrow \exists\ z \in B \subseteq A$ s.t $(x-\lambda) z = z(x-\lambda) = 1. \Rightarrow \lambda \in \rho_A(x)$.

(b) Let $\lambda \in \partial \sigma_B(x)$. Since $\sigma_B(x)$ is cpt $\lambda \in \sigma_B(x)$. and $\exists$ a sequence $\lambda_n \in \rho_B(x)$ s.t $\lambda_n \to \lambda$. By (a) $\lambda_n \in \rho_A(x)$. To show $\lambda \in \sigma_A(x)$. If this was not the case then, $(x-\lambda_n) \to (x-\lambda) \in \mathcal{G}(A)$. Then $(x-\lambda_n)^{-1} \to (x-\lambda)^{-1}$ but, $\lambda_n \in \rho_B(x)$ and B is a Banach subalgebra $\Rightarrow$ $(x-\lambda)^{-1} \in B$.

Prop Let $1, x \in B \subseteq A$ be inclusions of unital C^*-algs. Then $\sigma_A(x) = \sigma_B(x)$. In particular, if $A_0 = C^*(1,x)$ then $\sigma_A(x) = \sigma_{A_0}(x)$.

Proof:- Need to show $\rho_A(x) \subset \rho_B(x)$. Thus if λ is s.t $x-\lambda$ has inverse in A, the inverse should fall in B. In view a Sp. Mapping can assume $\lambda = 0$.

To show:- $1, x \in B \subseteq A$ $x \in \mathcal{G}(A) \Rightarrow x^{-1} \in B$.

Case 1 $x = x^*$. Let $A_0 = C^*(1,x)$. Note $\sigma_{A_0}(x) \subseteq \mathbb{R}$. Since a closed subset of $\mathbb{R}$ viewed as a subset of $\mathbb{C}$ is its own bdry so, noting $A_0 \subseteq B \subseteq A$ it follows

$$\sigma_A(x) \subseteq \sigma_B(x) \subseteq \sigma_{A_0}(x) = \partial \sigma_{A_0}(x) \subseteq \partial \sigma_B(x) \subseteq \partial \sigma_A(x) \subseteq \sigma_A(x)$$

so all equalities.

Case 2 $x \in \mathcal{G}(A) \Rightarrow x^* \in \mathcal{G}(A)$. $x^*x \in \mathcal{G}(A)$. By Case (i) $(x^*x)^{-1} \in B$. Let $y = (x^*x)^{-1}x^* \in B$. Then $yx = 1$. But x was inv. in A. Deduce that $y = x^{-1}$.

<u>Ex</u> A C^*- alg. $x \in A$. TFAE:-

(i) x is normal.

(ii) $C^*\{x\}$ is commutative.

(iii) If $x = x_1 + ix_2$ denotes the Cartesian decomposition then $x_1 x_2 = x_2 x_1$.

<u>Continuous Functional Calculus</u> :-

<u>Thm</u> Let $1, x \in A$ be normal and define $A_0 = C^*(1, x)$. Then exists a unique isometric $*$ - isomorphism $\Phi : C(\sigma(x)) \to A_0$ by $\Phi(f) = f(x)$ with the property that $f_1(x) = x$, when $f : \sigma(x) \to \mathbb{C}$ by $f_1(z) = z$.

<u>Proof</u>:- Write $\Sigma = \sigma(x)$. Note that $\hat{x} = \Gamma_{A_0}(x) : \hat{A} \to \Sigma$ is cont and surjective. We claim that $\hat{x}$ is a homeomorphism. Since $\hat{A}$, Σ are both cpt enough to check $\hat{x}$ is injective. Let $\varphi_1, \varphi_2 \in \hat{A}_0$ be s.t $\hat{x}(\varphi_1) = \hat{x}(\varphi_2)$. Write

$D = \left\{ \sum_{i,j=0}^{n} \alpha_{ij}\, x^i (x^*)^j : n \in \mathbb{N}, \alpha_{ij} \in \mathbb{C} \right\}$. and note that D is dense $*$ - subalg. of A_0. Clearly $\varphi_i(x^*) = \overline{\varphi_i(x)}$

$\therefore$ $\varphi_1|_D = \varphi_2|_D$. By density $\varphi_1 = \varphi_2$.

<u>Now</u> write $\Phi : C(\Sigma) \to A_0$ by

$$\Phi(f) = \Gamma_{A_0}^{-1}(f \circ \hat{x}).$$

Φ is linear, algebraic. Check that Φ preserves $*$.

<u>Now</u> $\|\Phi(f)\| = \|\Gamma_{A_0}^{-1}(f \circ \hat{x})\| = \|f \circ \hat{x}\| = \sup_{\varphi \in \hat{A}} |f \circ \hat{x}(\varphi)|$

$= \sup_{\lambda \in \sigma(x)} |f(\lambda)| = \|f\|.$

~~(let $y = \Gamma_{A_0}^{-1}(f \circ \hat{x})$.~~ ~~$\Gamma_{A_0}(y) = f \circ \hat{x}$~~

$\hat{y} = f \circ \hat{x}$ = ~~$\varphi(y) = \varphi(x)$ $\forall \varphi \in \hat{A}$~~

Φ hom $\Rightarrow$ $\Phi(1) = 1$ $\Rightarrow$

<u>Also</u> $\Phi(f_1) = x$. $\Rightarrow$

Again $D' = \{ f : f(z) = \sum_{i,j=0}^{n} \alpha_{ij} z^i \bar{z}^j \quad n \in \mathbb{N},\ \alpha_{ij} \in \mathbb{C} \}$ is dense in $C(\Sigma)$ by Stone Wierstrass. Since any cont $*$-hom of $C(\Sigma)$ is determined by its values on D' and hence on $f_0 = 1$, $f_1 = id$ we are done. $\square$.

<u>Moral</u> Any thing true for cont. functions should be true for normal (self-adjoints) in a C^*-alg.

<u>TFAE</u> ① x is normal and $\sigma(x) \subseteq \mathbb{R}$.

② x is self-adjoint.

<u>Pf</u> (i) $\Rightarrow$ (ii) By functional calculus we have $C^*(x) \cong C(\sigma(x))$ in which x corresponds to the identity function f_1 on $\sigma(x)$.

If $\sigma(x) \subseteq \mathbb{R}$ $x = x^*$ as $f_1 = \bar{f_1}$.

(ii) $\Rightarrow$ (i) $x = x^*$. So x normal. Nth we have show $\varphi(x) \in \mathbb{R}$ $\forall$ $\varphi \in \hat{A}_0$. $\Rightarrow$ $\sigma(x) \subseteq \mathbb{R}$.

<u>TFAE</u> (i) u normal and $\sigma(u) \subseteq \mathbb{T}$

(ii) u is unitary ie $u^* u = u u^* = 1$.

<u>TFAE</u> (i) p normal and $\sigma(p) \subset \{0,1\}$

(ii) p is a projection ie, $p = p^* = p^2$.

<u>Proof</u> Similar.

<u>Prop.</u> $x \in A$, TFAE:-

(i) x normal and $\sigma(x) \subseteq [0, \infty)$

(ii) x is positive i.e $\exists$ a self adjoint $y \in A$ s.t $x = y^2$.

<u>Pf</u> :- (i) $\Rightarrow$ (ii) Let $f(t) = t^{1/2}$ on $[0, \infty)$. Work $y = f(x)$ via functional calculus. $\Rightarrow$ $x = y^2$ as Φ was hom.

(ii) $\Rightarrow$ (i) :- If $f \in C(X)$, X cpt Haus dorff $\Rightarrow$ $\sigma(f) = f(X)$ (must try). Use function calculus for y for the function $f(t) = t^2$ on $\sigma(y)$.

Prop (Square Roots) $x \in A$ is positive. TFAE:

(i) $\exists$ a unique positive element $y \in A$ s.t $x = y^2$, $y \in C^*(\{x\})$.

and y is given by the function calculus of $f(t) = t^{1/2}$.

This is called the unique positive square root of x

and written as $y = x^{1/2}$.

Proof:- Existence of y is clear. as $\sigma(x) \subseteq [0,\infty)$.

Note the function $\sqrt{t}$ is a limit of polynomials

without constant term. It follows $y \in C^*(\{x\})$.

Uniqueness let $z \in A$ be positive and $x = z^2$.

It follows that if $A_0 = C^*\{z\}$ then $x \in A_0$. and

consequently, $y \in C^*(\{x\}) \subseteq A_0$. But now if $X = \sigma(z)$

then $A_0 \cong C(X)$ and under this isomorphism

both y and z corresponds non negative functions whose

squares are same. It follows $z = y$.

Prop:- let $x = x^* \in A$. Then $\exists$ a unique decomposition $x = x_+ - x_-$

where $x_\pm$ are positives which satisfy $x_+ x_- = 0$.

Proof:- let $A_0 = C^*(1, x) = C^*(\{x\})$. On the $\sigma(x) \subseteq \mathbb{R}$

define functions $f_\pm = \max(0, \pm t)$. Note that $f_\pm \geq 0$

and $f_+(t) - f_-(t) = t$ and $f_+(t) f_-(t) = 0$ $\forall\, t \in \sigma(x)$.

Also note $f_\pm(0) = 0$ so by functional calculus $f_\pm(x)$

$\in A_0$ and has the desired properties.

Uniqueness:- Suppose $x = x_+ - x_-$ is a decomp. of the desired

type. Then $x_- x_+ = x_-^* x_+^* = (x_+ x_-)^* = 0$. So x_+ and x_-

are commuting positives in A. It follows $A_0 = C^*(x_+, x_-)$ is abelian

and $x, x_\pm \in A_0$. and so $f_\pm(x) \in C^*(\{x\}) \subseteq A_0$.

Fact:- X cpt T_2 space, $g_i : X \to \mathbb{R}$ non-negative cts.

s.t $f = g_1 - g_2$ $g_1 g_2 = 0$ $\Rightarrow$ $g_1 = \max(f, 0)$, $g_2 = \max(-f, 0)$.

Finish the proof.

Thm $x = x^*$. $f \in C(\sigma(x))$ Then $\sigma(f(x)) = f(\sigma(x))$.

Proof:- Under the cannonical isomorphism from
$C(\sigma(x)) \mapsto A_0 = C^*(\{x\})$
$g(t) = t \to x$. and $f(x)$ corresponds to f. Then seen before, $\sigma(f) =$ Range f. Spectrum is invariant under $*$-isomorphism and enough to compute $\sigma(f(x))$ relative to A_0.

Thm Every element in a C^*-alg is a linear combination of atmost (exactly) four unitaries.

Proof:- ~~Enough~~ ~~to~~ ~~consider~~ Assume $x = x^*$ first.
and let $y = \frac{x}{\|x\|}$ so that $\|y\| = 1$.
$\therefore \sigma(y) = \hat{y}(\hat{A}_0)$ when $\hat{A_0} = C^*(\{y\})$.
$\Rightarrow \sigma(y) \subseteq [-1, 1]$.
On $C(\sigma(x))$ choose functions $f(t) = t + i\sqrt{1-t^2}$
then f and $\bar{f}$ are unitaries and $t = \frac{1}{2}(f(t) + \bar{f}(t))$.
let $f(y) = u$. $\therefore u^* u = uu^* = 1$ and
$y = \frac{1}{2}(u + u^*)$ by function calculus.
~~Thm~~ $x = \|x\| y = \frac{\|x\|}{2}(u + u^*)$.
In general write $x = x_1 + i x_2$ Cartesian decomp. and apply the above to real and imaginary parts.

A fact need to prove this)

Prop If x is normal then $r(x) = \|x\|$.

Proof:- let x be s.a. So $\|x^{2n}\| = \|(x^n)^* x^n\| = \|x^n\|^2$
By induction :- $\|x^{2^n}\| = \|x\|^{2^n}$ $\forall n$. $\Rightarrow r(x) = \|x\|$.
When $x^*x = xx^*$ then, by above (x^*x sa)
$\|x\|^2 = \|x^*x\| = r(x^*x) \leq r(x^*) r(x) = r(x)^2 \leq \|x\|^2$
$\Rightarrow r(x) = \|x\|$.

Prop A unital commutative Banach alg. and $x, y \in A$.

Then $\sigma(xy) \subseteq \sigma(x)\,\sigma(y)$ and $\sigma(x+y) \subseteq \sigma(x) + \sigma(y)$.

Moreover, $r(xy) \leq r(x)\, r(y)$, $r(x+y) \leq r(x) + r(y)$.

Proof:- (a) Let $\lambda \in \sigma(xy)$ then $xy - \lambda$ lies inside a maximal ideal of A, which is then the kernel of unique complex hom. φ. and $\lambda = \varphi(xy) = \varphi(x)\,\varphi(y)$.

Note $x - \varphi(x)1$, $y - \varphi(y)1 \in \ker \varphi$. So $\varphi(x) \in \sigma(x)$, $\varphi(y) \in \sigma(y)$. Then $\lambda \in \sigma(x)\,\sigma(y)$.

(b) Let $\lambda \in \sigma(x+y)$. $\exists$ multiplicative linear functional φ s.t $\lambda = \varphi(x) + \varphi(y)$; $\varphi \in \sigma(x)$, $\varphi(y) \in \sigma(y)$

$\Rightarrow$ $\lambda \in \sigma(x) + \sigma(y)$

The spectral radius formula is then automatic.

Thm $x = x^*$. Then at least one of the nos $\pm \|x\|$ belongs to $\sigma(x)$.