Banach Algebras :-

Def^n A normed algebra is a normed space A_0 with the following structure: There exists a well defined multiplication on A_0 i.e a map $A_0 \times A_0 \to A_0$ denoted by $(x,y) \to xy$ which satisfies — for all $x, y, z \in A_0$ and $\alpha \in \mathbb{C}$,

(i) $(xy)z = x(yz)$ (associative)

(ii) $(\alpha x + y) z = \alpha xz + yz$, $z(\alpha x + y) = \alpha zx + zy$ (distributive)

(iii) (sub multiplicative norm) $\|xy\| \leq \|x\| \|y\|$.

Def^n A Banach algebra is a normed algebra A such that A is a Banach space. A is called unital if $\exists$ an element $1 \in A_0$ s.t $1x = x1 = x \; \forall x$.

Ex :- (i) $B(X)$: all bounded functions on a set with sup norm and pointwise multiplication.

(ii) Sub algebras (closed) of Banach algebras with inherited structure.

Just as we can complete a metric space to obtain complete metric spaces, so can complete normed algebras to Banach algebras.

(iii) $C_c(\mathbb{R})$ - compactly supported cts function is normed alg. with natural structure

$C_0(\mathbb{R})$ - completion of $C_c(\mathbb{R})$ a Banach alg.

(iv) X locally compact Hausdorff.

$C_c(X)$ - normed alg.

$C_0(X)$ - Banach alg. completion.

(v) $\ell^1(\mathbb{Z})$ with $\|f\|_1 = \sum |f(n)|$ and multiplication

$(f * g)(n) = \sum_m f(m) g(n-m)$ convolution product.

(vi) $C_c(\mathbb{R})$ with convolution product.

$$(f * g)(x) = \int f(y)\, g(x-y)\, dy$$

and $\|f\|_1 = \int |f|\, dx$.

Completion of $C_c(\mathbb{R})$ is $L^1(\lambda)$.

Can give more general examples with groups.

(vii) Exer. Wiener algebra

$$A(\mathbb{T}) := \{ f : [-\pi, \pi] \to \mathbb{C} \mid \sum_n |\hat{f}(n)| < \infty \}$$

where $\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-int}$. So $f \in A(\mathbb{T})$

Fourier series converge uniformly,

$\Rightarrow$ $f \in C(\mathbb{T})$. In fact

$$\|f\|_\infty \leq \|f\|$$

Pointwise product, gives Banach alg structure.

Ex In all the examples above check if the Banach alg is abelian and which are unital.

(viii) X - Banach algebra.

$L(X) = \{ T : X \to X$ bdd linear map$\}$.

is Banach alg. under composition. Check commutativity.

Propⁿ A_0 - normed alg. Consider $A_0^+ = A_0 \oplus \mathbb{C}$ as vector space

i.e $A_0^+ = \{ (x, \alpha) : x \in A_0, \alpha \in \mathbb{C} \}$ and

with product $(x, \alpha) \cdot (y, \beta) = (xy + \alpha y + \beta x, \alpha\beta)$

$\|(x,\alpha)\| = \|x\| + |\alpha|$.

Then (i) A_0^+ is unital normed alg.

(ii) the map $A_0 \ni x \longrightarrow (x, 0) \in A_0^+$ is isometric isomorphism.

(iii) A_0 is Banach algebra $\Leftrightarrow$ A_0^+ is.

Propⁿ A is a Banach alg. $\mathcal{J} \subseteq A$ is a closed ideal ($x, y \in \mathcal{J}$, $z \in A$, $\alpha \in \mathbb{C}$, $\alpha x + y$, xz, $zx \in \mathcal{J}$).

Then $A/\mathcal{J}$ is a Banach alg. with prod. $(x+\mathcal{J}) \cdot (y+\mathcal{J}) = xy + \mathcal{J}$.

Convention: The norm of 1 in a unital Banach alg is assumed to be 1.

Henceforth assume A is unital and $A \neq 0$ i.e. $1 \neq 0$.

Let $\mathcal{G}(A)$ denote the group of all invertible elements of A.

Prop (i) If $x \in \mathcal{G}(A)$, define $L_x : A \to A$ by $L_x(y) = xy$ then $L_x \in L(A)$ and L_x is invertible $\Leftrightarrow$ $x \in \mathcal{G}(A)$

(ii) $x \in \mathcal{G}(A)$, $y \in A$ then $xy \in \mathcal{G}(A) \Leftrightarrow yx \in \mathcal{G}(A) \Leftrightarrow y \in \mathcal{G}(A)$.

(iii) $\{x_1, \ldots, x_n\} \subset A$ s.t. $x_i x_j = x_j x_i$ $\forall\, i,j$, then the products $x_1 \cdots x_n \in \mathcal{G}(A) \Leftrightarrow$ each $x_j \in \mathcal{G}(A)$

(iv) $x \in A$ and $\|1 - x\| < 1$ then $x \in \mathcal{G}(A)$ and

$$x^{-1} = \sum_{n=0}^{\infty} (1-x)^n.$$

Particularly, $\lambda \in \mathbb{C}$ and $|\lambda| > \|x\|$, then $(x - \lambda)$ is invertible and

$$(x-\lambda)^{-1} = -\sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}. \quad \text{— } (1)$$

(v) $\mathcal{G}(A)$ is an open set and $x \to x^{-1}$ is a homeomorphism of $\mathcal{G}(A)$ to itself.

Proof:- (iv) note that $\sum_{n=0}^{\infty} \|1 - x\|^n < \infty$, then $\sum_{n=0}^{\infty} (1-x)^n$ defines an element s of A.

let $s_n = \sum_{i=0}^{n} (1-x)^i$. Then note that $(1-x)s_n = s_n(1-x)$
$= s_{n+1} - 1$.

Then $(1-x)s = s(1-x) = s - 1$ in the limit.

i.e. $xs = sx = 1$.

(v) note that (iv) shows that a open ball around 1 lie in $\mathcal{G}(A)$. It is then not difficult to transport this local property to the entire of $\mathcal{G}(A)$.

'$\varnothing$'

Def.^{n} (i) $\mathcal{A}$ unital Banach alg. $x \in \mathcal{A}$.

The spectrum $\sigma(x) = \{ \lambda \in \mathbb{C} : (x-\lambda) \text{ is not invertible} \}$.

and spectral radius

$$r(x) = \sup \{ |\lambda| : \lambda \in \sigma(x) \}.$$

(this is the radius of the smallest disc centred at 0 containing the spectrum).

(ii) The resolvent of x denoted by $\rho(x)$ is

$$\rho(x) = \mathbb{C} \setminus \sigma(x) = \{ \lambda \in \mathbb{C} : (x-\lambda) \text{ has inverse} \}.$$

and the map $\rho(x) \ni \lambda \xrightarrow{R_x} (x-\lambda)^{-1} \in \mathcal{G}(\mathcal{A})$ resolvent function.

Note By (iv) before $r(x) \leq \|x\|$. and by (v) $\rho(x)$ is open and R_x is cont.

Consequently, $\sigma(x)$ is compact.

Prop let $x \in \mathcal{A}$.

(a) $\lim\limits_{|\lambda| \to \infty} \| R_x(\lambda) \| = 0$.

(b) $R_x(\lambda) - R_x(\mu) = (\lambda-\mu) R_x(\lambda) R_x(\mu)$, $\lambda, \mu \in \rho(x)$.

(c) R_x is weakly analytic (i.e. $\varphi \circ R_x$ - analytic for $\varphi \in \mathcal{A}^*$)

and $\lim\limits_{|\lambda| \to \infty} \varphi \circ R_x(\lambda) = 0$.

Proof:- (a) For $|\lambda| > \|x\|$, $\| R_x(\lambda) \| \leq \dfrac{C}{|\lambda|}$, C constant.

(b) $(\lambda-\mu) R_\lambda(x) R_\mu(x) = (\lambda-\mu)(x-\lambda)^{-1}(x-\mu)^{-1}$

$$= R_x(\lambda) \left((x-\mu) - (x-\lambda) \right) R_x(\mu)$$

$$= R_x(\lambda) - R_x(\mu).$$

(c) $\varphi \in \mathcal{A}^*$. If $\mu \in \rho(x)$ and λ is close to μ then $\lambda \in \rho(x)$.

then $\lim\limits_{\lambda \to \mu} \dfrac{\varphi \circ R_x(\lambda) - \varphi \circ R_x(\mu)}{\lambda - \mu} = \varphi\left((R_x(\mu))^2 \right)$.

Then $\varphi \circ R_x$ is weak analytic. The rest follows from (a).

Thm A unital, $x \in A$, $\sigma(x) \neq \varphi$.

Proof:- Let $\sigma(x) = \varphi$. Then $\rho(x) = \mathbb{C}$. Then R_x is weakly entire and $\varphi \circ R_x \to 0$ as $|\lambda| \to \infty$ $\forall \varphi$.

By Liouville's thm $\varphi \cdot R_x(\lambda) = 0$ $\forall \lambda$. $\forall \varphi$. But

By Hahn-Banach theorem $R_x(\lambda) = 0$ $\forall \lambda$.

$R_x(\lambda) \subseteq \mathcal{G}(A)$ and this is absurd.

Thm (i) $x \in A$. Let $p(z) = \sum_{k=0}^{n} a_k x^k$, $p(x) = \sum_{k=0}^{n} a_n x^k$ $(x^0 = 1)$.

Then $\mathbb{C}[z] \ni p(z) \longrightarrow p(x) \subseteq A$ is a homomorphism whose range is the subalgebra generated by 1 and x.

(ii) (Spectral Mapping):- $\sigma(p(x)) = p(\sigma(x)) = \{p(\lambda) : \lambda \in \sigma(x)\}$.

Proof:- Skipped

Theorem (Spectral Radius Formula) $x \in A$ then $r(x) = \lim_{n \to \infty} \|x^n\|^{1/n}$.

Proof:- Fix $\varphi \in A^*$. Let $F = \varphi \circ R_x$. By def^n this is analytic on the complement of the disc $\{\lambda : |\lambda| \leq r(x)\}$.

On the other hand for $|\lambda| > \|x\|$, $(x-\lambda)^{-1} = -\sum_{n=0}^{\infty} \frac{x^n}{\lambda^{n+1}}$ $\Rightarrow$ that $F(\lambda) = -\sum_{n=0}^{\infty} \frac{\varphi(x^n)}{\lambda^{n+1}}$ is Laurent series expansion of F.

Since F vanishes at ∞, F is analytic at ∞ and so the Laurent series expansion is valid for $\{\lambda : |\lambda| > r(x)\}$.

Fix λ s.t $|\lambda| > r(x)$. Then $\lim_n \frac{\varphi(x^n)}{\lambda^n} = 0$ $\forall \varphi$.

By principle of uniform boundedness $\exists$ constant K s.t

$$\|x^n\| \leq K |\lambda|^n. \quad \forall n.$$

$\Rightarrow$ $\|x^n\|^{1/n} \leq K^{1/n} |\lambda|$. Let $|\lambda| \downarrow r(x)$.

$\Rightarrow$ $\|x^n\|^{1/n} \leq K^{1/n} r(x)$. i.e $\overline{\lim} \|x^n\|^{1/n} \leq r(x)$.

By spectral mapping $\sigma(x^n) = \{\lambda^n : \lambda \in \sigma(x)\}$.

$\Rightarrow$ $r(x) = r(x^n)^{1/n} \leq \|x^n\|^{1/n}$. $\forall n$.

$\Rightarrow$ $r(x) \leq \underline{\lim}_n \|x^n\|^{1/n}$. $\square$.

From now on let A be commutative Banach alg.

1 **Prop** let $x \in A$. TFAE:-

(a) x is not invertible.

(b) $\exists$ a maximal ideal $\mathfrak{g} \subseteq A$ st $x \in \mathfrak{g}$.

Pf (ii) $\Rightarrow$ (i) obvious.

(i) $\Rightarrow$ (ii) x not invertible. let $x \neq 0$ else trivial.

let $\mathfrak{g} = \{ ax : x \in A\}$ is ideal in A. $\mathfrak{g} \neq 0$ as $x \in \mathfrak{g}$.

$\mathfrak{g} \neq A$ as $1 \notin \mathfrak{g}$. By Zorn's lemma absorb A inside a maximal ideal. $\blacksquare$.

Thm:- (Gelfand - Mazur Theorem):- TFAE:- (A unital abelian).

(a) A is a division alg (every non zero element in invertible)

(b) A is simple (no proper ideals in A).

(c) $A = \mathbb{C}1$.

Proof:- Easy (use previous thm).

Homomorphism :-

A commutative Banach alg. A complex homomorphism $\varphi : A \to \mathbb{C}$ is a linear map s.t $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi \neq 0$.

Let $\hat{A} = \{ \varphi : A \to \mathbb{C}$ complex hom.$\}$. is called the spectrum of A.

For $x \in A$, Let $\hat{x} : \hat{A} \to \mathbb{C}$ by $\hat{x}(\varphi) = \varphi(x)$.

(Observe that for a unital Banach alg. $\varphi \neq 0 \iff \varphi(1) = 1$).

Lemma :- (a) The $\varphi \to \ker\varphi$ is a bijective correspondence between $\hat{A}$ and maximal ideals of A.

(b) $\hat{x}(\hat{A}) = \sigma(x)$.

(c) (Since $\|1\| = 1$) For $\varphi : A \to \mathbb{C}$ the following are equivalent

(i) $\varphi \in \hat{A}$, (ii) $\varphi \in A^*$, $\|\varphi\| = 1$, $\varphi(xy) = \varphi(x)\varphi(y)$.

Proof:- (a) Let $\varphi \in \hat{A}$. and $J = \ker \varphi$. Since φ is alg. hom onto $\mathbb{C}$, J is maximal ideal. Conversely, let J be maximal ideal. By Gelfand Mazur, $A/J = \mathbb{C}$. Let $q : A \to A/J$ be the quotient map. Then q is a complex. hom, $q \neq 0$, and $\ker q = J$. The bijective correspondence follows from the fact that two functionals with same kernel are multiples of each other. and if $\varphi \in \hat{A}$, then $A = \ker \varphi \oplus \mathbb{C}1$ as vector spaces and $\varphi(1) = 1$.

(b) $x \in A$ and $\varphi \in \hat{A}$. Then $(x - \varphi(x)1) \in \ker \varphi$. = max ideal. $\Rightarrow \varphi(x) \in \sigma(x)$. The reverse direction follows from (a) and ①.(previous page).

(c) $\varphi \in \hat{A}$. From (b) $|\varphi(x)| \leq r(x) \leq \|x\|$. $\Rightarrow \|\varphi\| \leq 1$; $\varphi(1) = 1$ $\Rightarrow \|\varphi\| = 1$. (other way is obvious).

Cor A unital Banach alg. $\varphi \in \hat{A} \Rightarrow \|\varphi\| \leq 1$ (non-unital).

Gelfand Transform:- Let A be abelian with spectrum $\hat{A}$. Then

(a) $\hat{A}$ is l.c Hausdorff space. (and is compact is $\hat{A}$ has 1).

(b) $\Gamma : A \to C_0(\hat{A})$ by $\Gamma(x) = \hat{x}$ is a contractive hom. of Banach algs.

Proof:- Note $\hat{A} \subseteq A_1^*$. Equip $\hat{A}$ with the w^*-top.
(note A_1^* is w^*-cpt Hausdorff space Alaoglu thm).

(a) $x, y \in A$, $K_{x,y} = \{\varphi \in A_1^* : \varphi(x)\varphi(y) = \varphi(xy)\}$. Then $K_{x,y}$ is w^*-closed. V is w^*-open $V = A_1^* \setminus \{0\}$. in A_1^*. Note $\hat{A} = \bigcap_{x,y \in A} K_{x,y} \cap V$. Use open subset of l.c. Hausdorff space is lc Hausdorff to say $\hat{A}$ is lc Hausdorff space. (If $1 \in A$ then $\hat{A} = \bigcap_{x,y} K_{x,y} \cap \{\varphi \in A_1^* : \varphi(1) = 1\}$ $\Rightarrow$ cpt Hausdorff).

(b) Clearly that $\Gamma(x) \in C_0(\hat{A})$ and check Γ is linear and multiplicative. (Only prove contractivity in unital case) then $C_0(\hat{A}) = C(\hat{A})$.

$$\|\Gamma(x)\| = \sup_{\varphi \in \hat{A}} |\hat{x}(\varphi)| = \sup_{\varphi \in \hat{A}} |\varphi(x)| = \sup_{\lambda \in \sigma(x)} |\lambda| = r(x) \leq \|x\|. \quad \square$$

Def^n A C^*-algebra is by def^n a Banach algebra A equipped with an involution $A \ni x \to x^* \in A$ which satisfy the following. For $x, y \in A$, $\alpha \in \mathbb{C}$,

(i) $(\alpha x + y)^* = \bar{\alpha} x^* + y^*$, (ii) $(xy)^* = y^* x^*$ (iii) $(x^*)^* = x$

(iv) $\| x^* x \| = \| x \|^2$.

x^* - is said to be the adjoint of x.

Example $C_0(X)$, X loc. cpt Hausdorff, $B(H)$ $\dim(H) \geq 2$, self-adjoint norm closed subalgebras of a C^*-algebra i.e $K(H)$, compacts.

Check that $\| x^* \| = \| x \|$, $1^* = 1$.

General facts

x is self-adjoint $x = x^*$

x is normal $x x^* = x^* x$.

x is unitary $x x^* = x^* x = 1$ (provided 1 exists).

x is projection $x = x^* = x^2$.

Any $x = \underbrace{\frac{x + x^*}{2}}_{s.a} + i \underbrace{\frac{x - x^*}{2i}}_{s.a}$ So A is a span of self-adjoints.

Lemma :- A not necessarily unital Banach alg. Then $e^x = \sum \frac{x^n}{n!}$ defines an element of A, such that $e^x \in \mathcal{G}(A)$. If x and y commute then $e^{x+y} = e^x e^y = e^y e^x$.

Then for $x \in A$, $\mathbb{R} \longrightarrow \mathcal{G}(A)$ by $x \longrightarrow e^{tx}$ is a continuous homomorphism of topological groups.

Gelfand Naimark Theorem :- Let A be a commutative C^* algebra with 1. Then the Gelfand transform $\Gamma : A \longrightarrow C(\hat{A})$ is a isometric $*$-algebra isomorphism.

Proof:- As noted before $\hat{A}$ is compact in w^*-topology. Then show Γ is linear, algebraic. To just show $\Gamma(x^*) = (\Gamma(x))^*$, $\| \Gamma(x) \| = \| x \|$. and Γ is surjective.

Start with $x = x^* \in A$.

Write $u_t = e^{itx}$ $\forall\ t \in \mathbb{R}$. Notice that by continuity of $*$

$$u_t^* = \sum_{n=0}^{\infty} \left(\frac{(itx)^n}{n!}\right)^* = \sum_{n=0}^{\infty} \frac{(-itx)^n}{n!} = u_{-t}.$$

Also check that $u_t^* u_t = u_t u_t^* = 1$ (u_t is 1-parameter group of unitaries).

$\Rightarrow$ $\|u_t\|^2 = \|u_t u_{-t}\| = 1.$

Let $\varphi \in \hat{A}$. $\|\varphi\| = 1$ $\Rightarrow$ $|\varphi(u_t)| \leq 1.$ $\forall t.$

φ is cont. and multiplicative $\Rightarrow$ $\varphi(u_t) = e^{it\varphi(x)}.$

$\Rightarrow$ $|e^{it\varphi(x)}| \leq 1$ $\forall\ t \in \mathbb{R}.$

$\Rightarrow$ $\varphi(x) \in \mathbb{R}.$

So $\Gamma(x) = \hat{x}$ is a real valued function.

Let $y \in A$. Write $y = x_1 + i x_2$ x_i's s.a.

$\therefore$ $\Gamma(y) = \Gamma(y_1) + i\,\Gamma(y_2)$

$\Rightarrow$ $\Gamma(y)^* = \Gamma(y_1) - i\,\Gamma(y_2) = \Gamma(y_1 - iy_2) = \Gamma(y^*).$

So Γ is $*$-alg hom.

Again if $x = x^* \in A$ then $\|x\|^2 = \|x^*x\| = \|x^2\|.$

By induction $\|x\|^{2^n} = \|x^{2^n}\|.$

$\therefore$ By spectral radius formula, $r(x) = \lim_n \|x^{2^n}\|^{1/2^n} = \|x\|.$

But note $\|\Gamma(x)\| = r(x) = \|x\|.$

In general, $\cancel{\|\Gamma(x)\|^2} = \|\Gamma(x^*x)\| = \|x^*x\|$

$\Rightarrow$ $\|\Gamma(x)\|^2 = \|\Gamma(x)^*\Gamma(x)\| = \|x\|^2.$

Surjectivity :- Note $\Gamma(A) \subseteq C(\hat{A})$ is a norm closed $*$ subalgebra which contains constants, conjugates and separates points. By Stone-Weierstrass $\Gamma(A) = C(\hat{A})$. $\square$.